COV886 Special Module in Algorithms: Computational Social Choice

Lecture 11

Fair Division of Indivisible Goods

March 26, 2022 | Rohit Vaish

Reminder about starting recording

Vaccine COVID-19
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The Model

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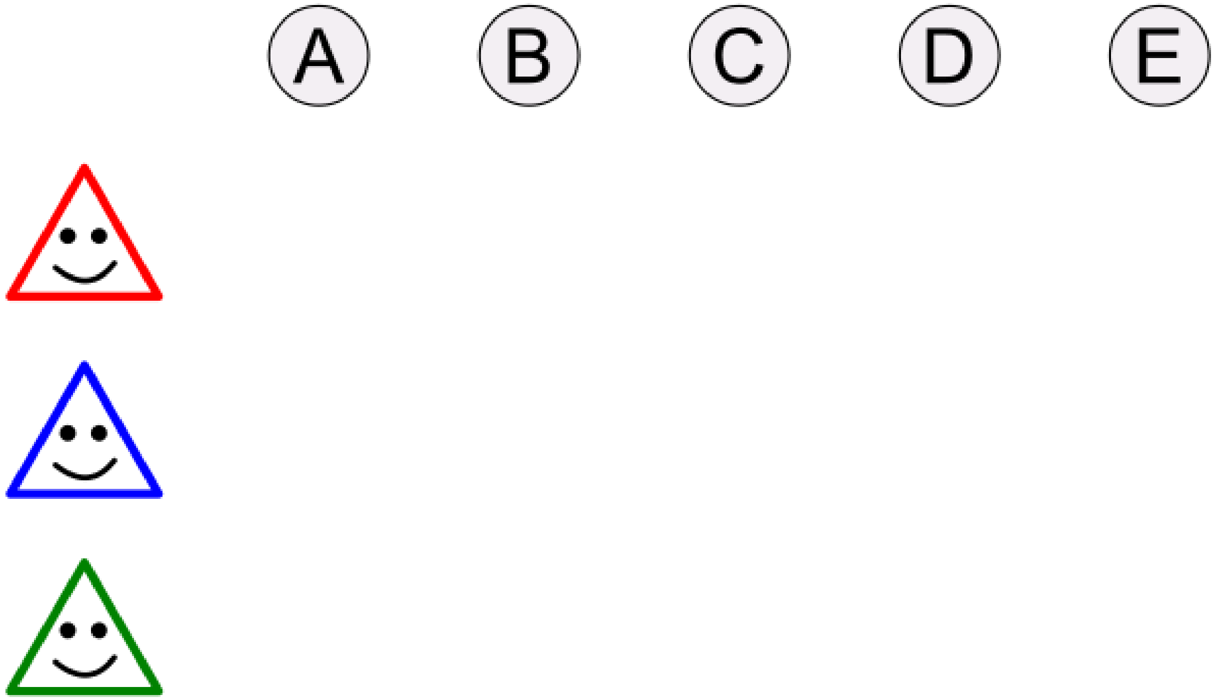

The Model

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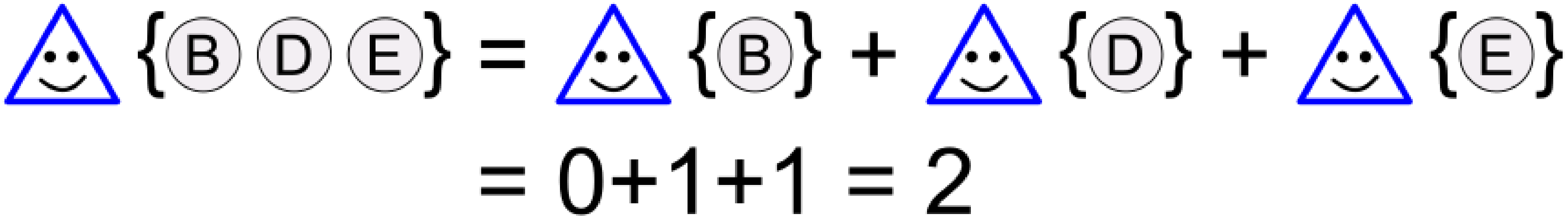
Additive valuations



Envy-Freeness [Gamow and Stern, 1958; Foley, 1967]

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Each agent prefers its own bundle over that of any other agent.

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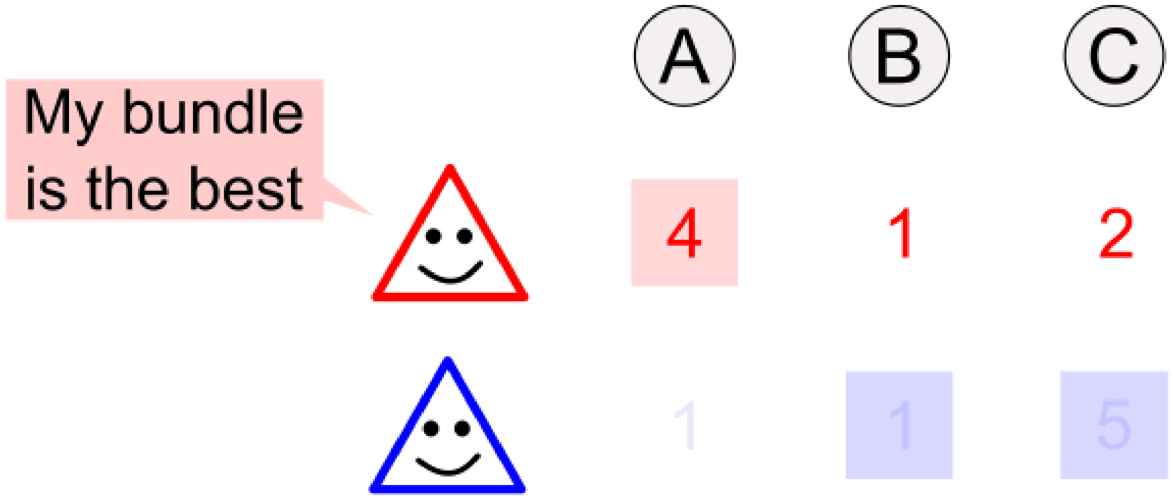

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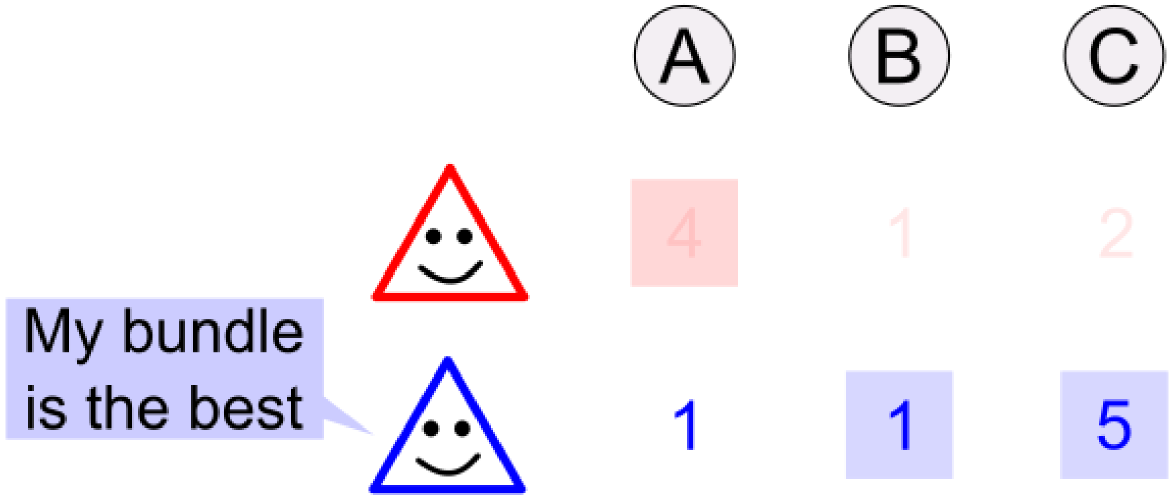

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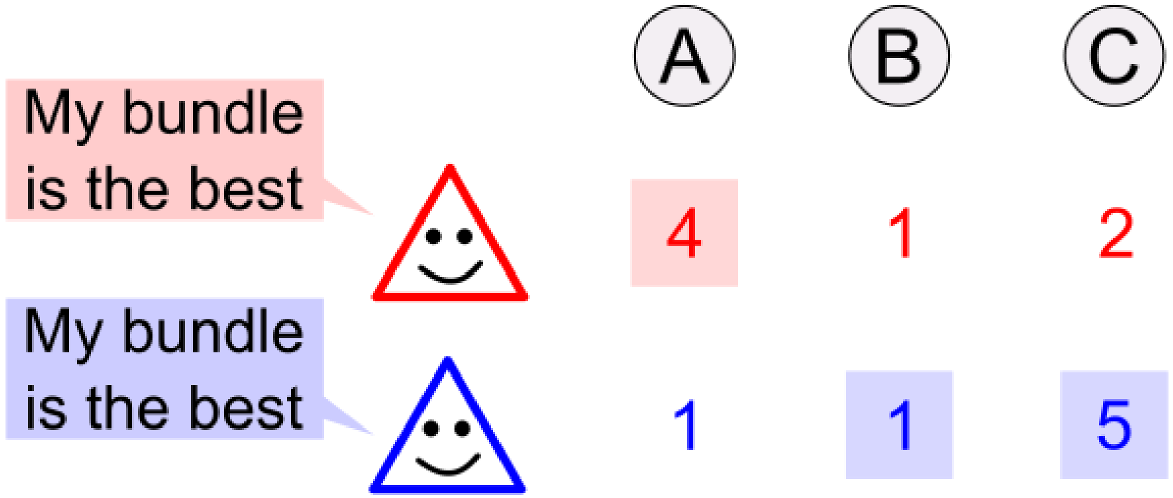

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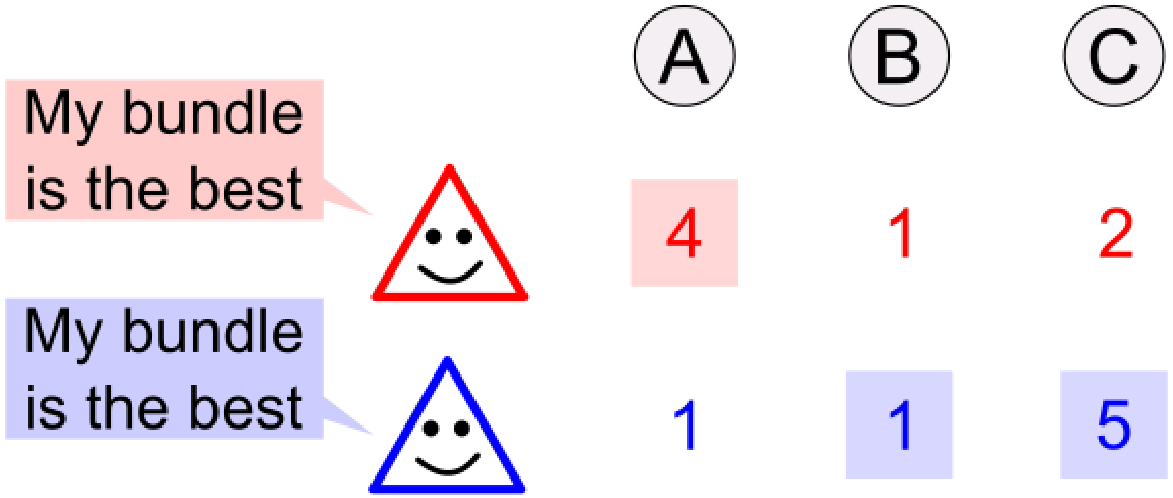

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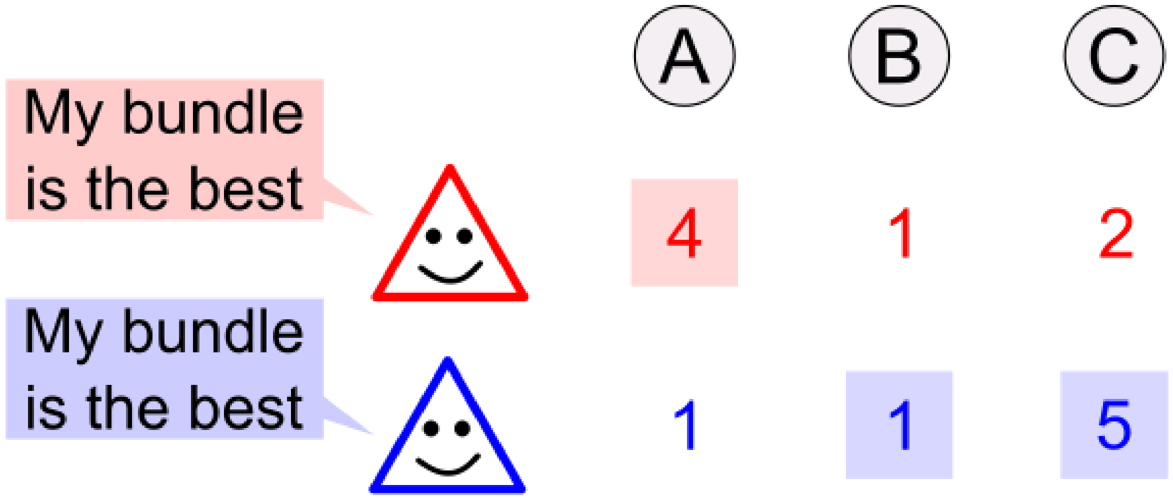

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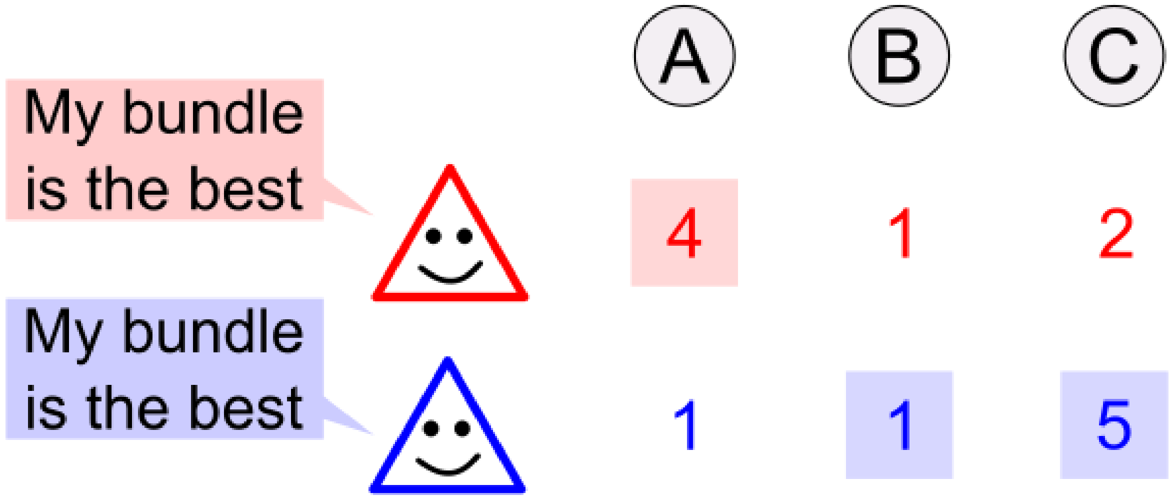

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- Not guaranteed to exist (two agents, one good)
- Checking whether an EF allocation exists is NP-complete

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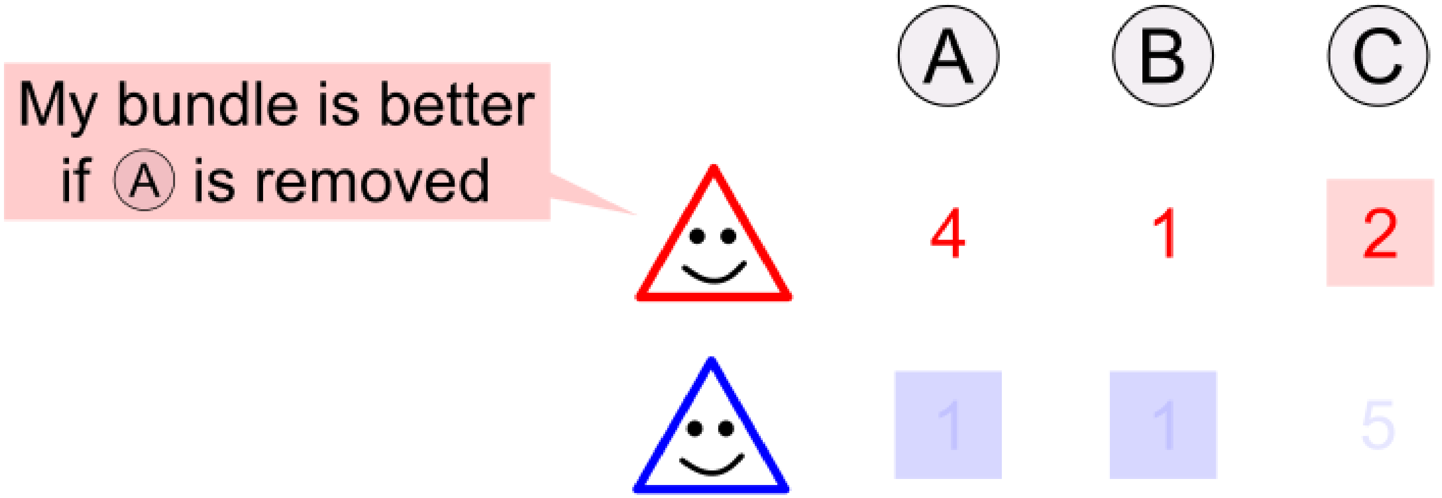

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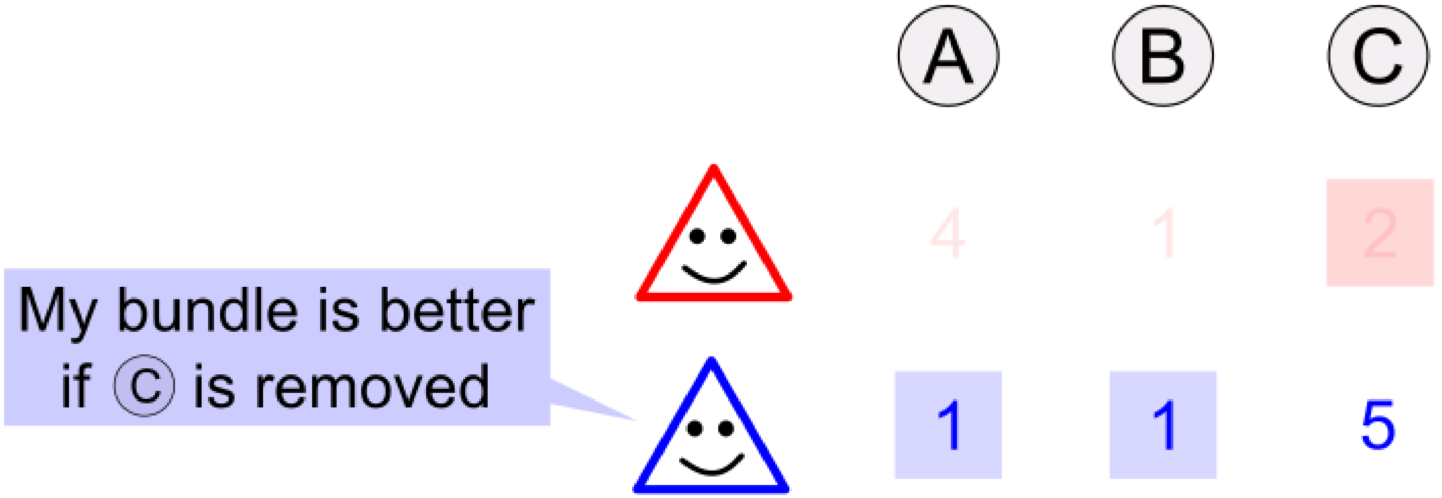

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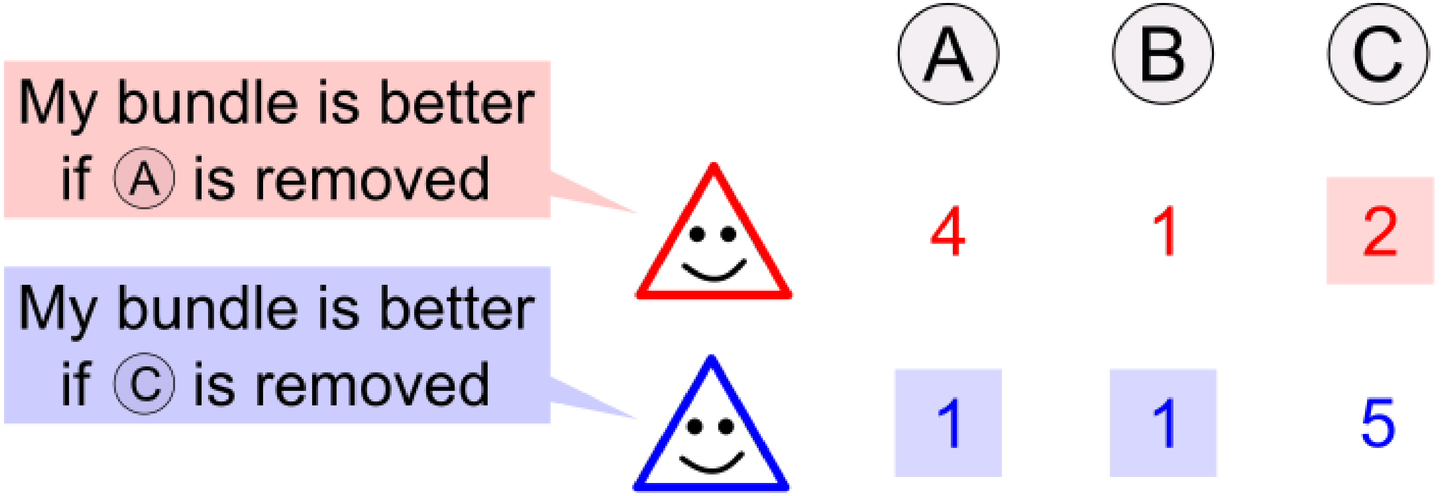

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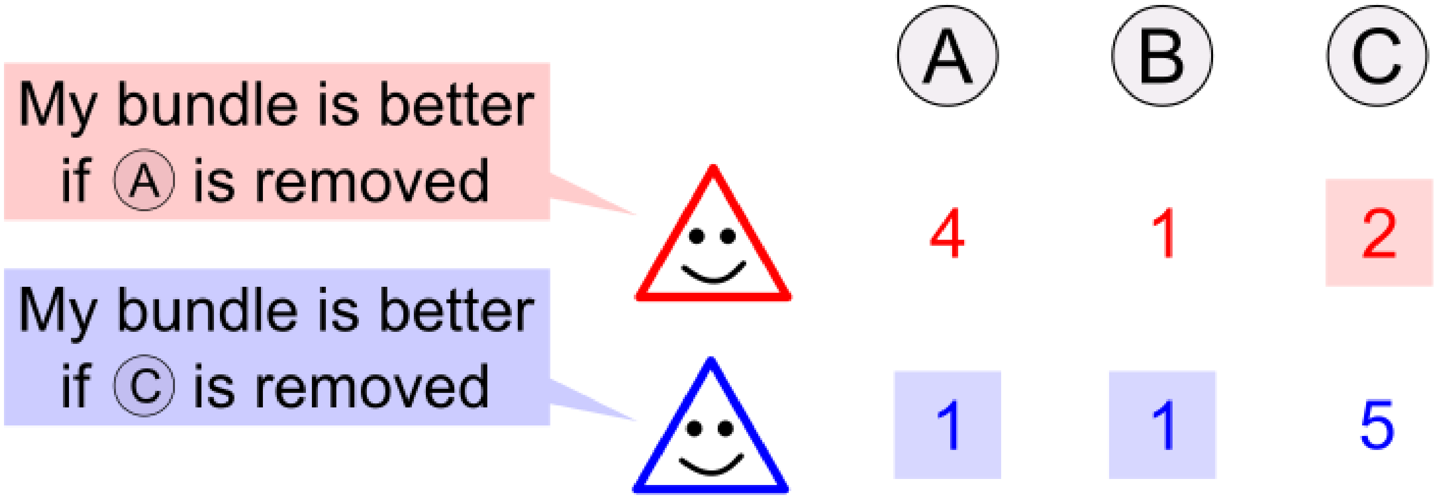

Allocation $A = (A_1, \ldots, A_n)$ is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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	A	B	C
Red agent	4	1	2
Blue agent	1	1	5

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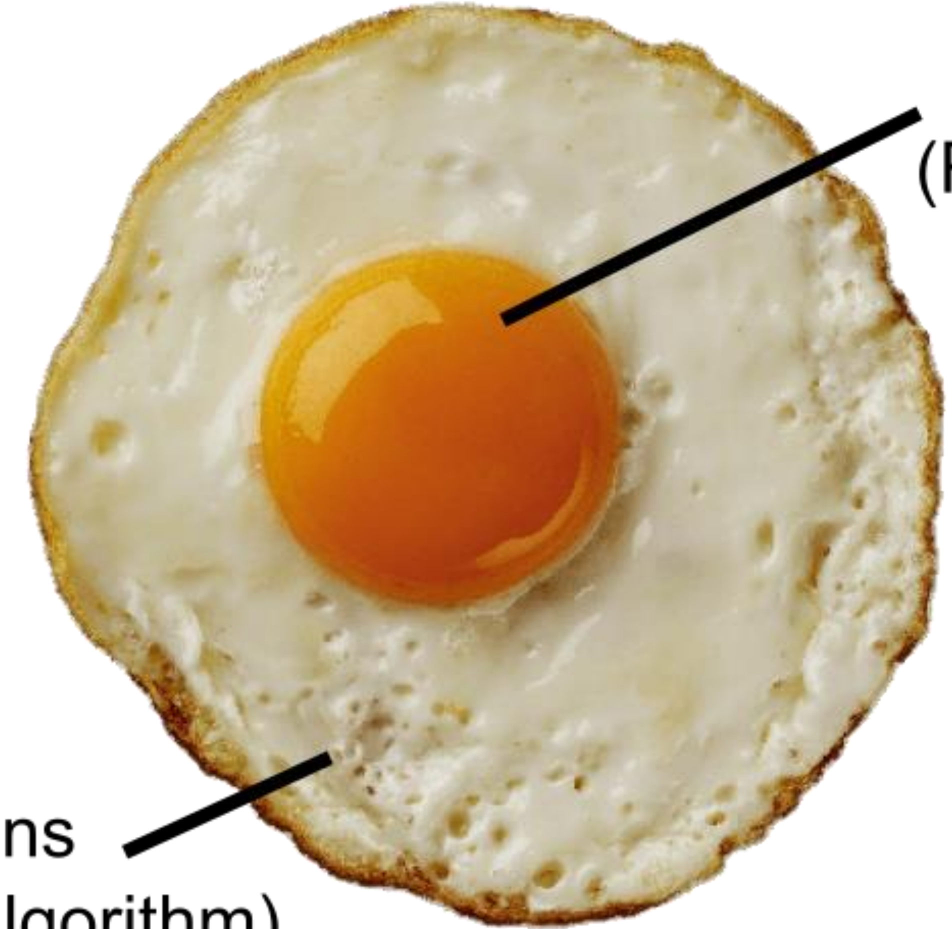
Guaranteed to exist and efficiently computable

Coming Up

Algorithms for finding an EF1 allocation



Additive valuations
(Round-robin algorithm)

Monotone valuations
(Envy-cycle elimination algorithm)

Round-robin algorithm

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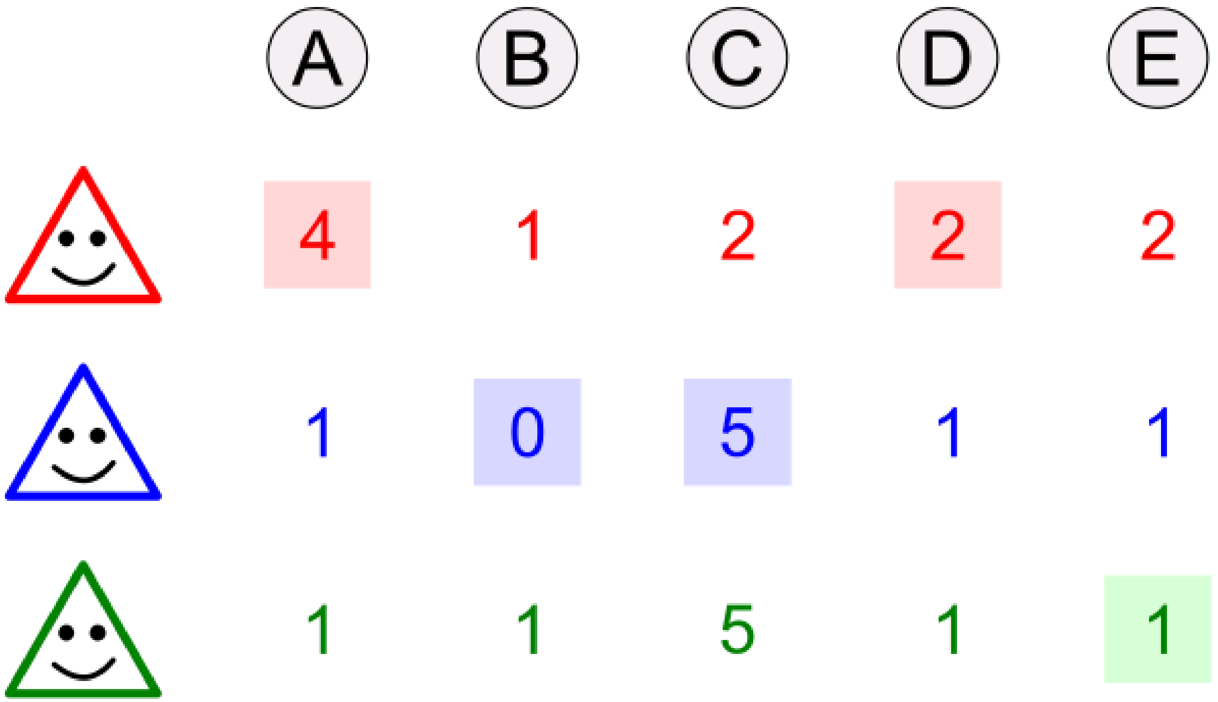

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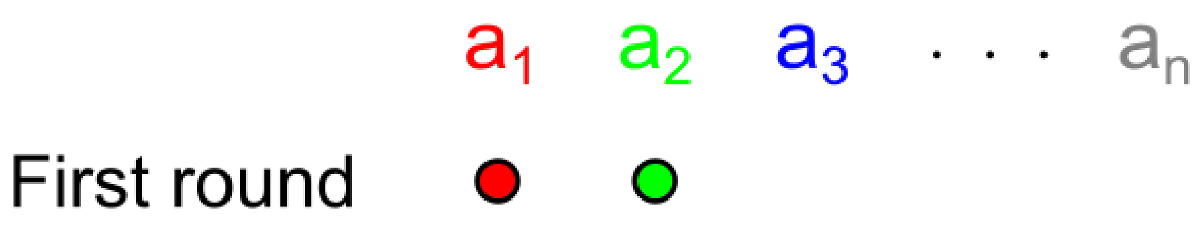

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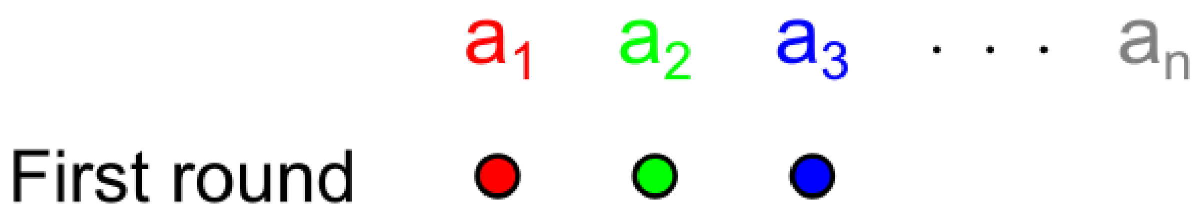

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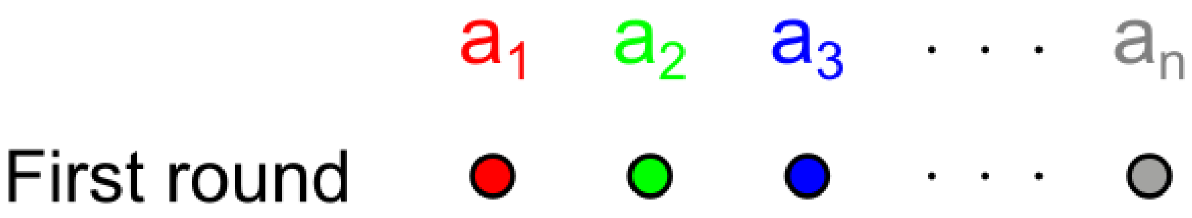

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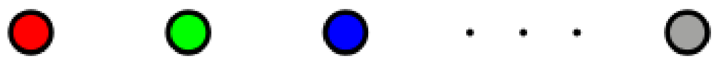
First round 

Second round

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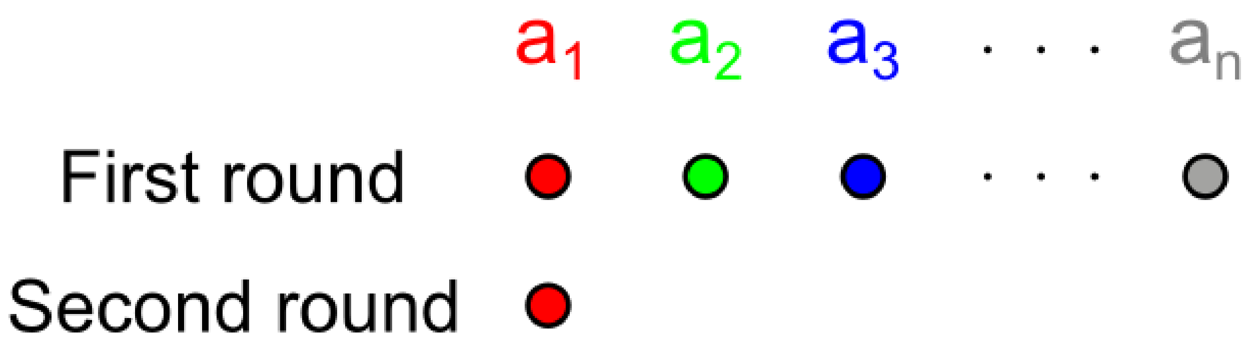

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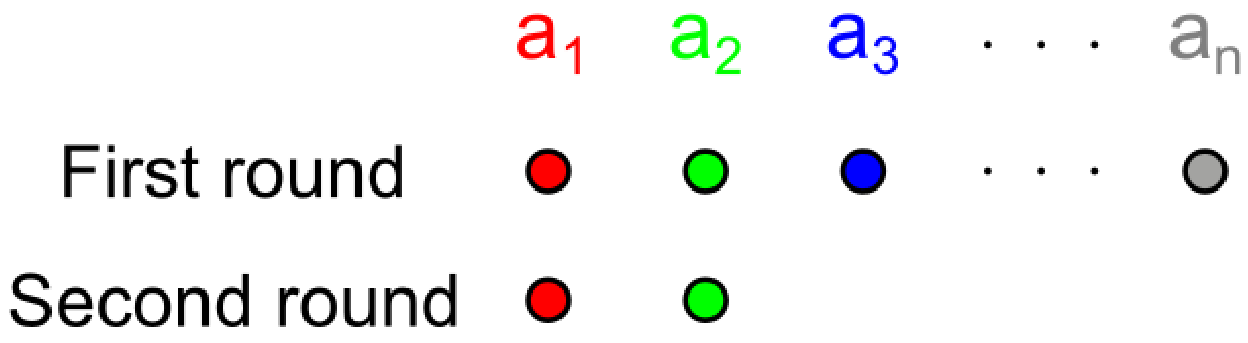

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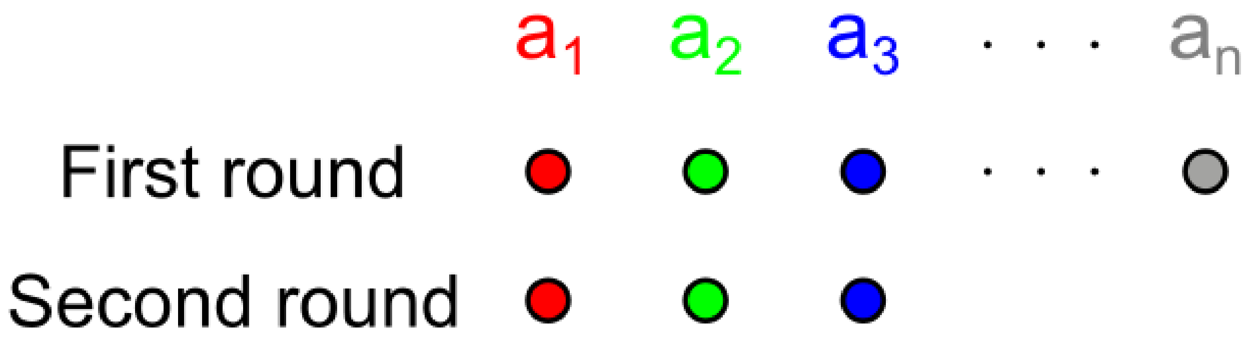

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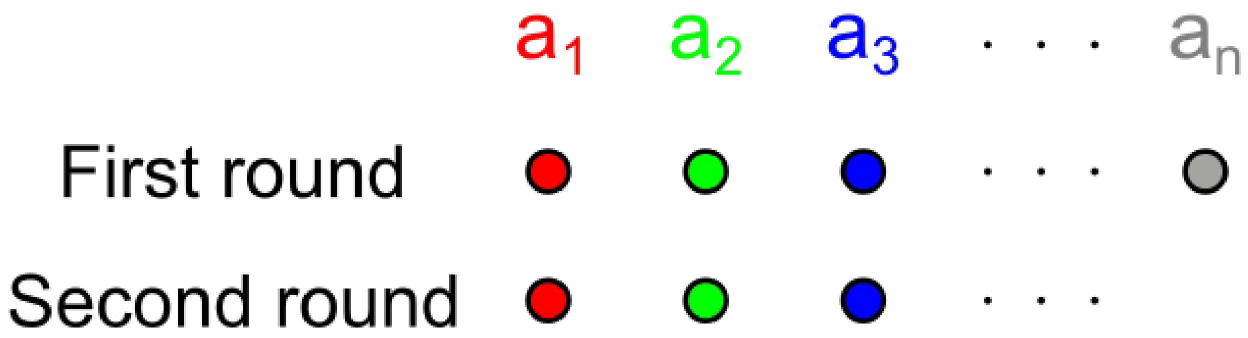

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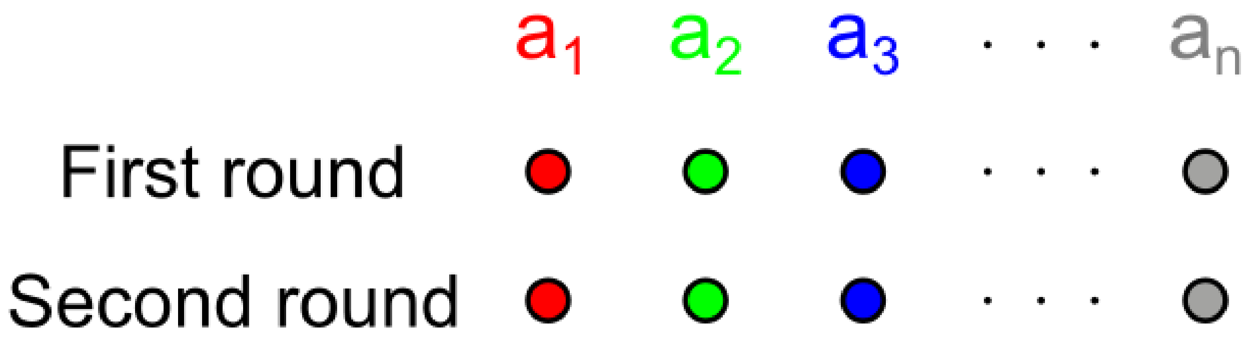

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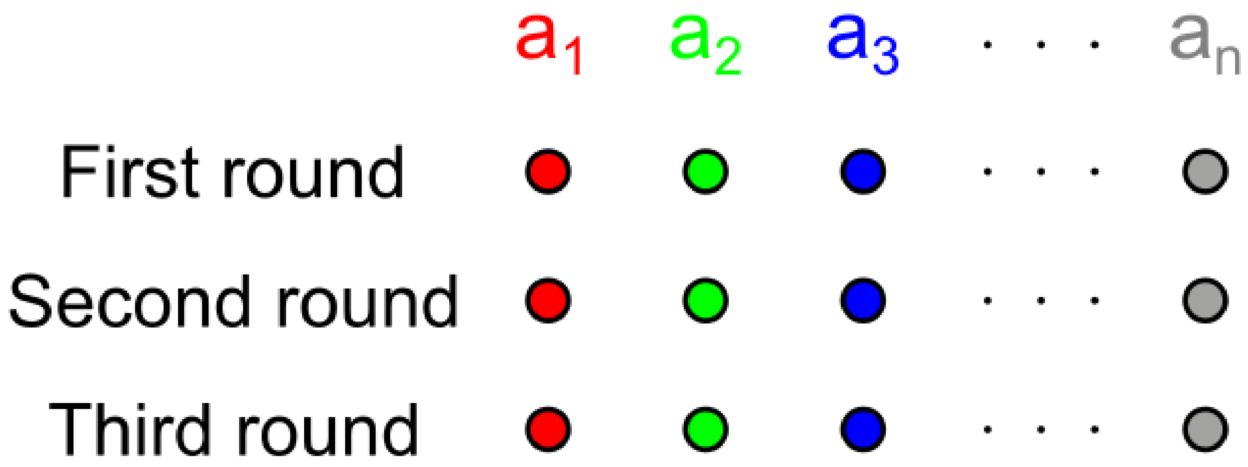

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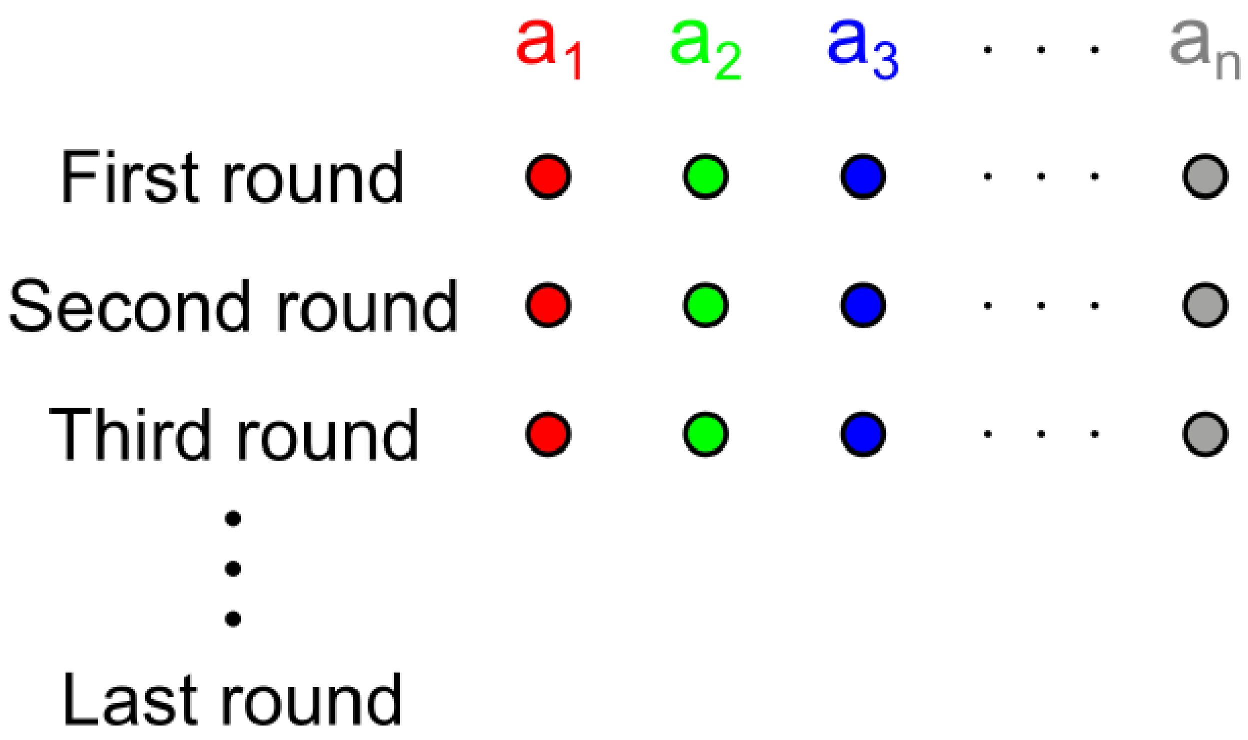

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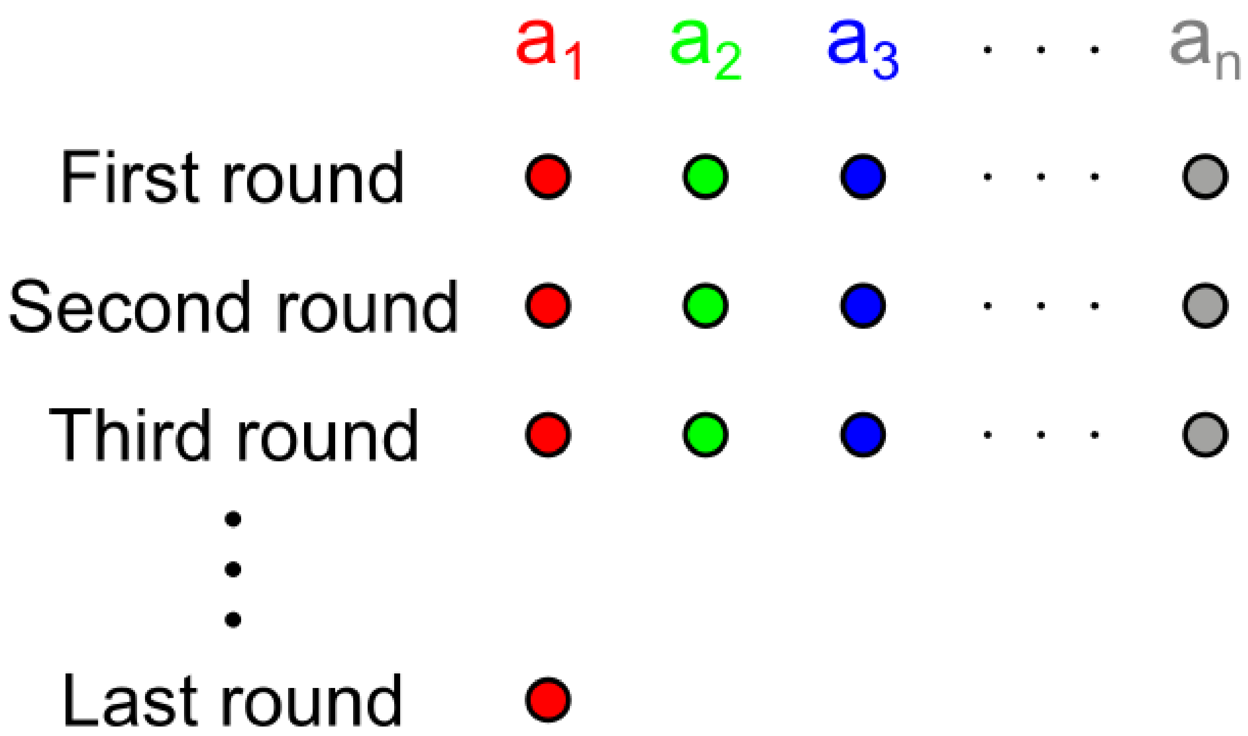

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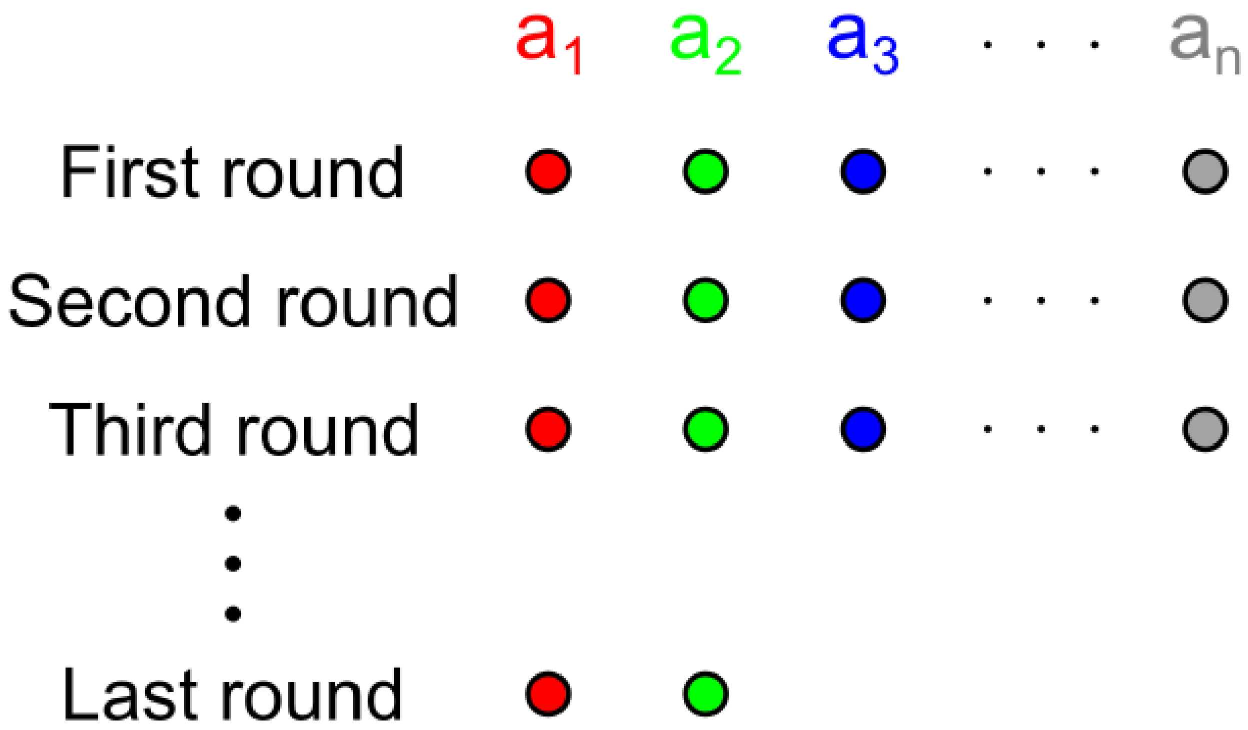

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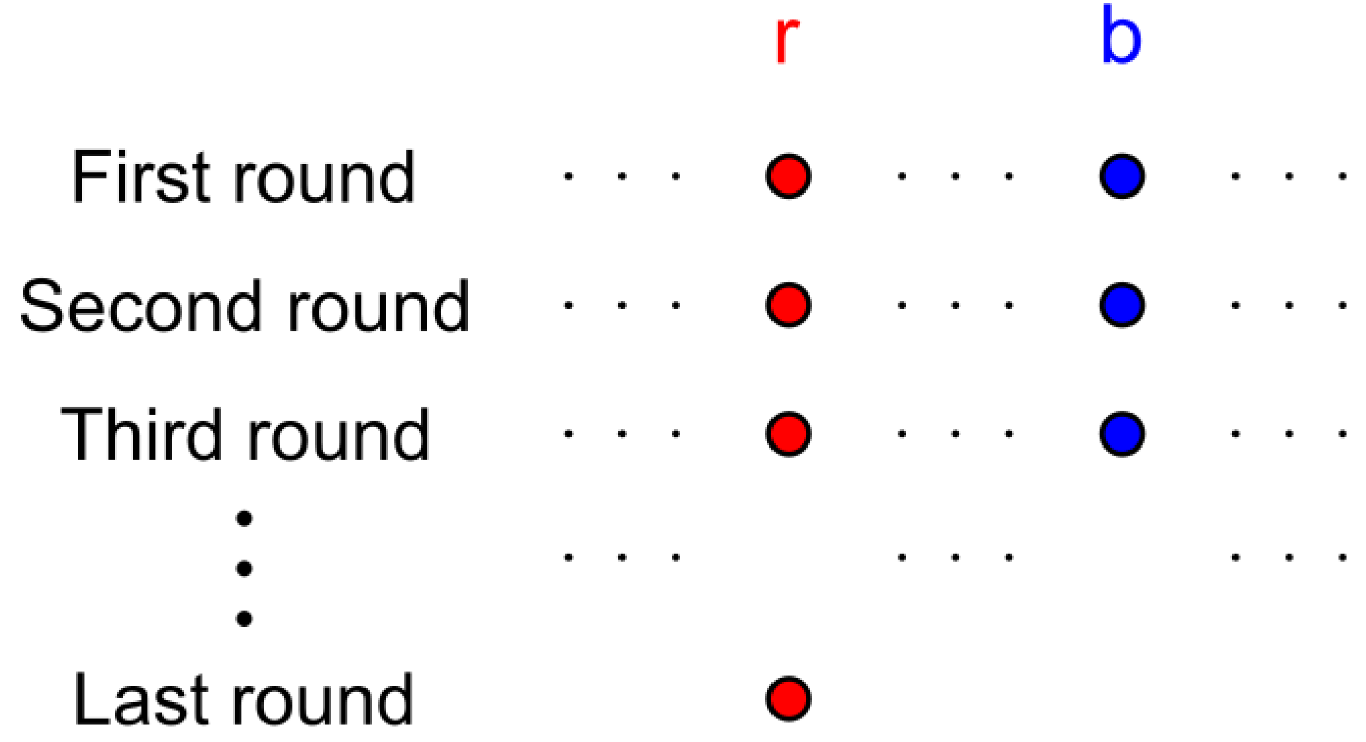

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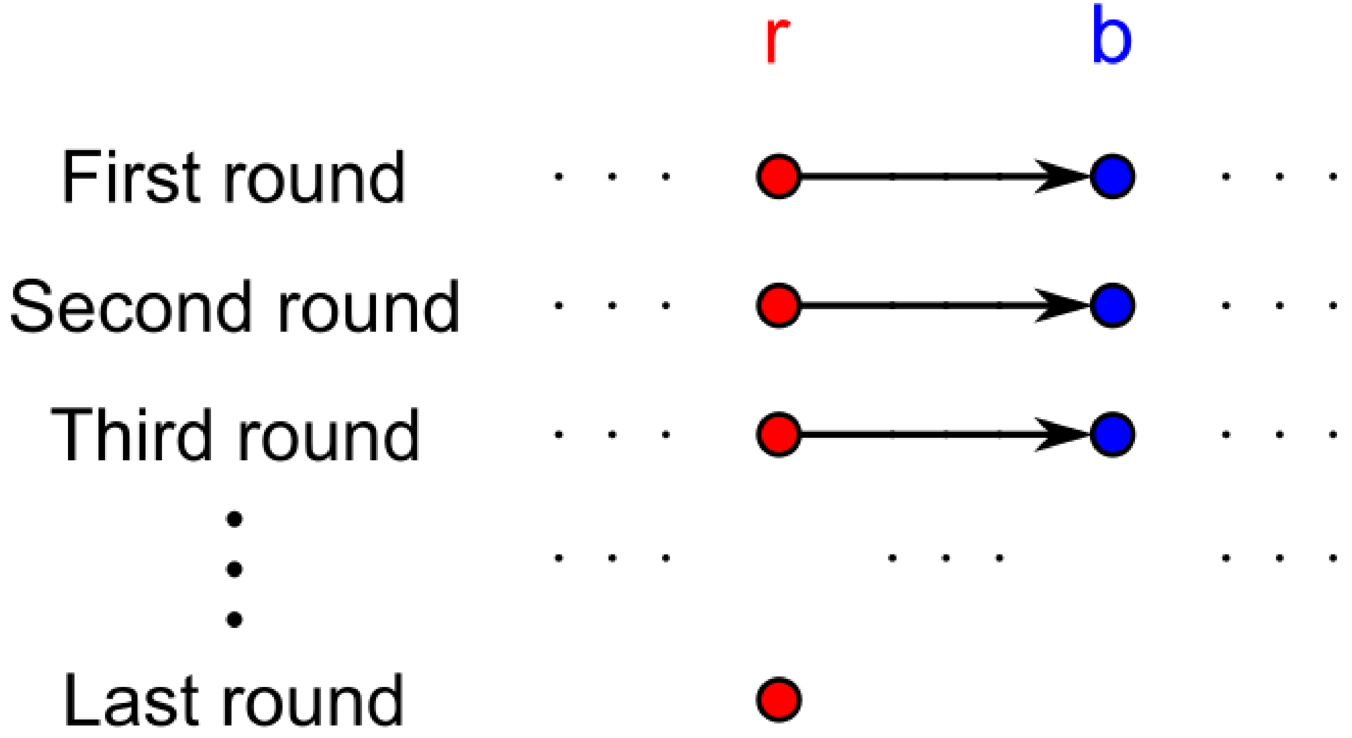

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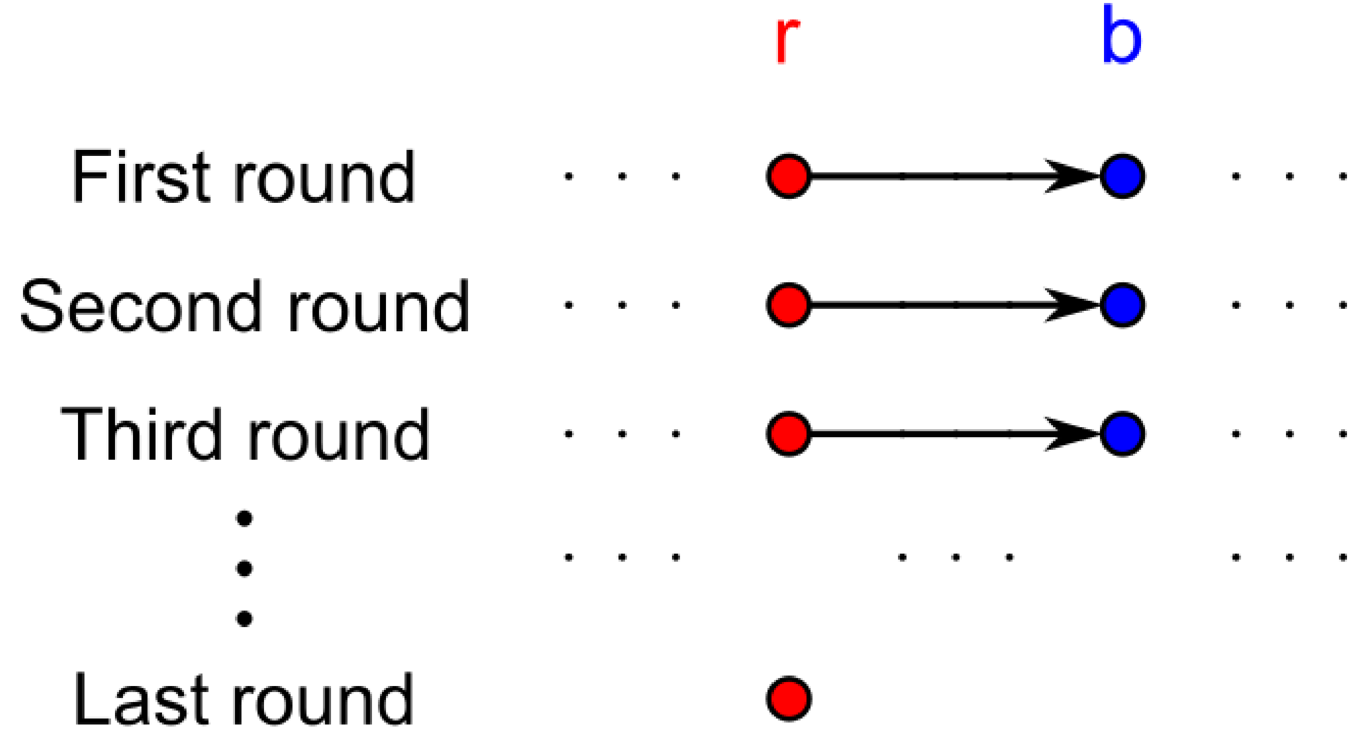

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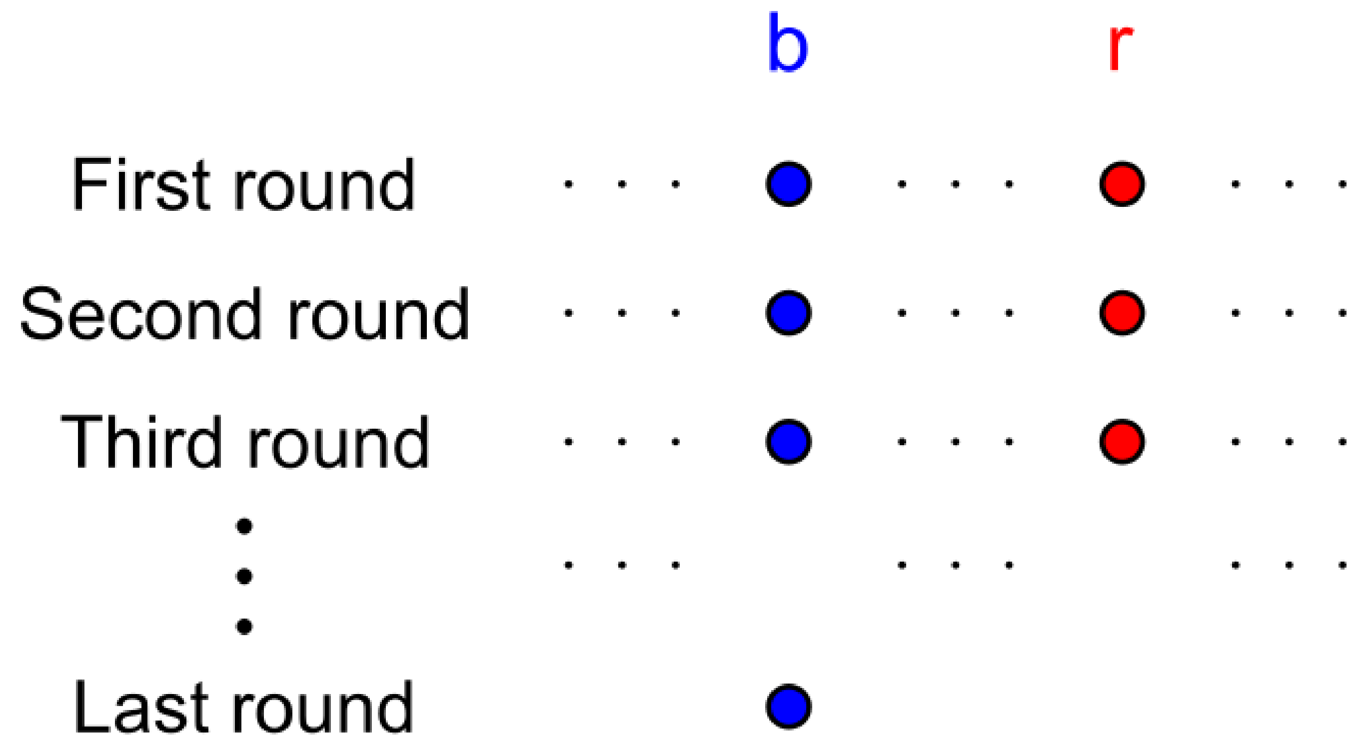

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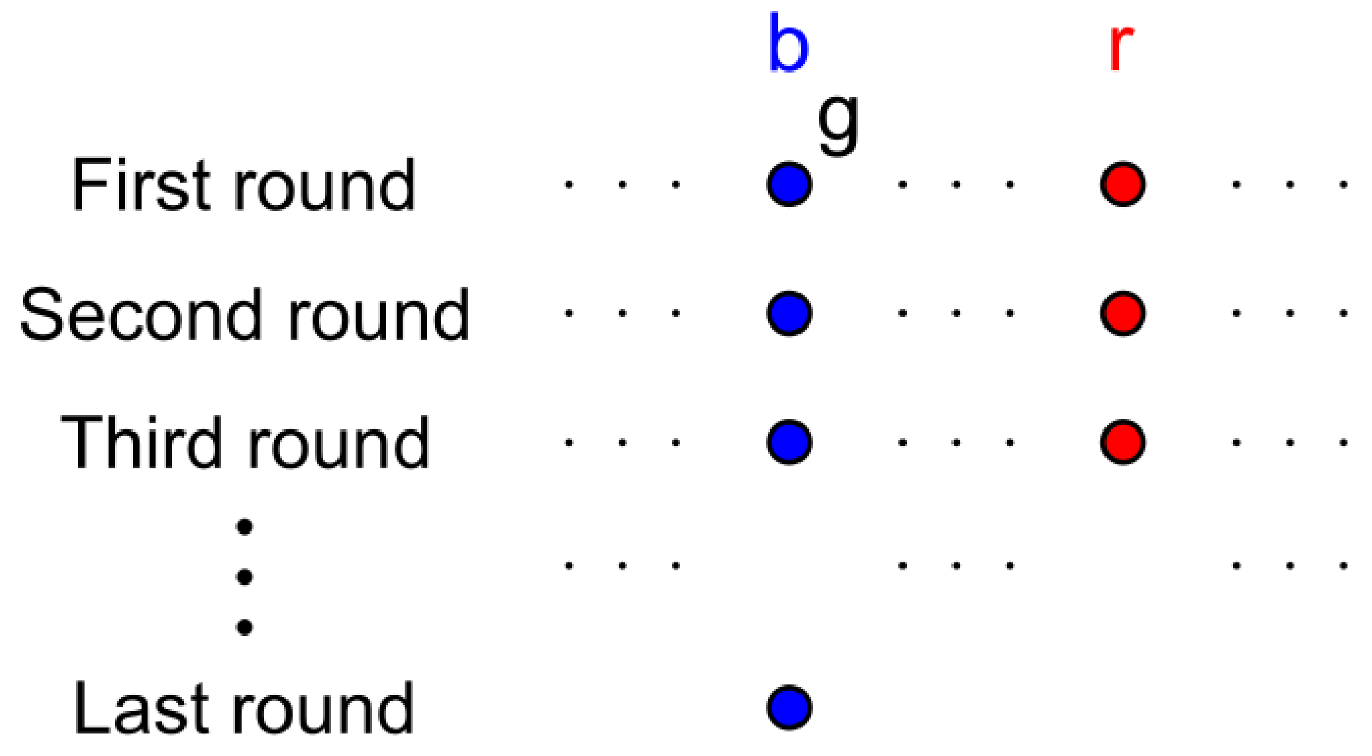

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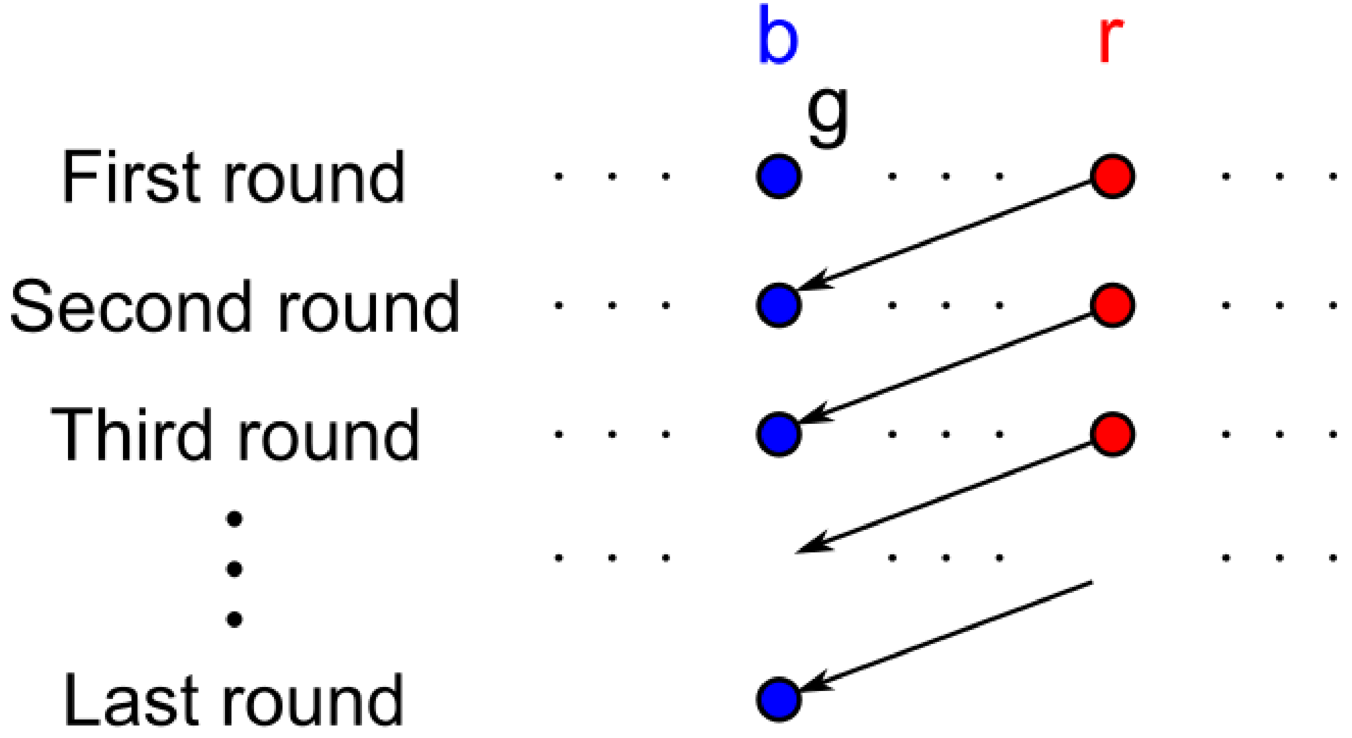

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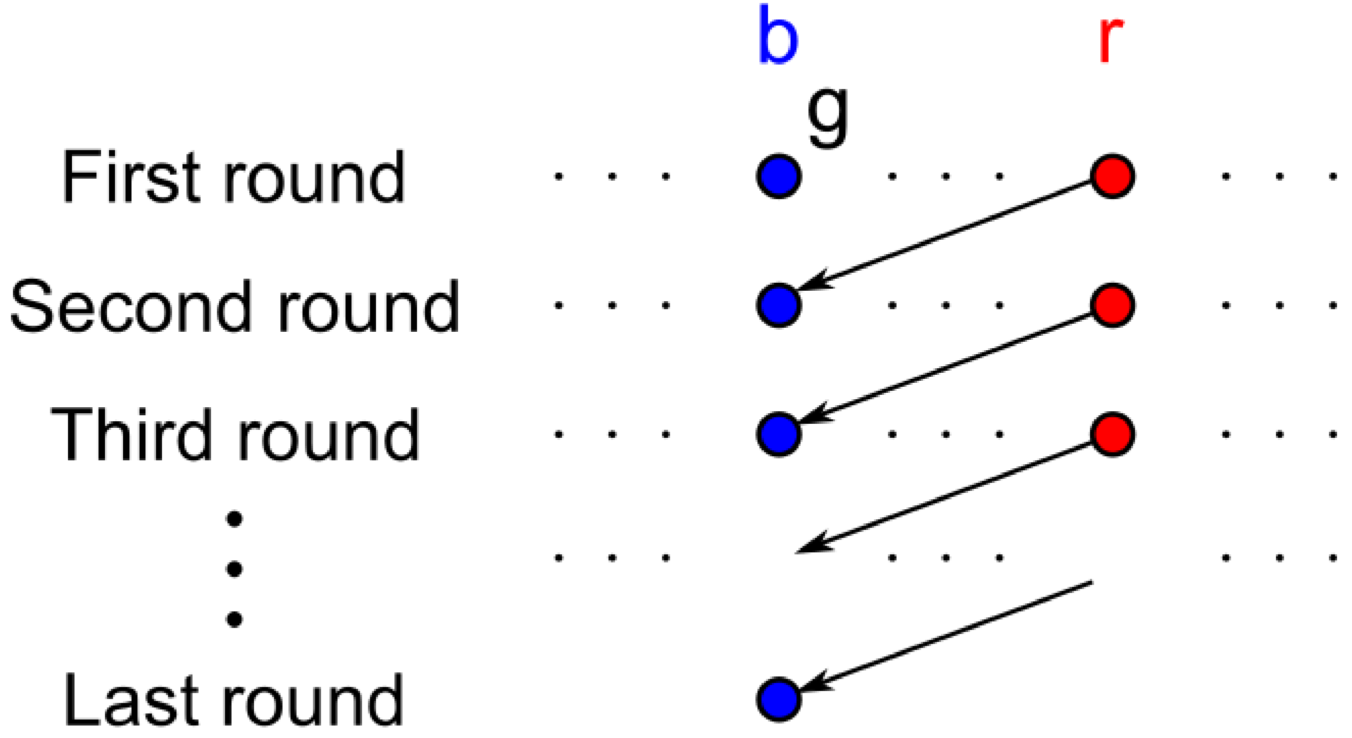

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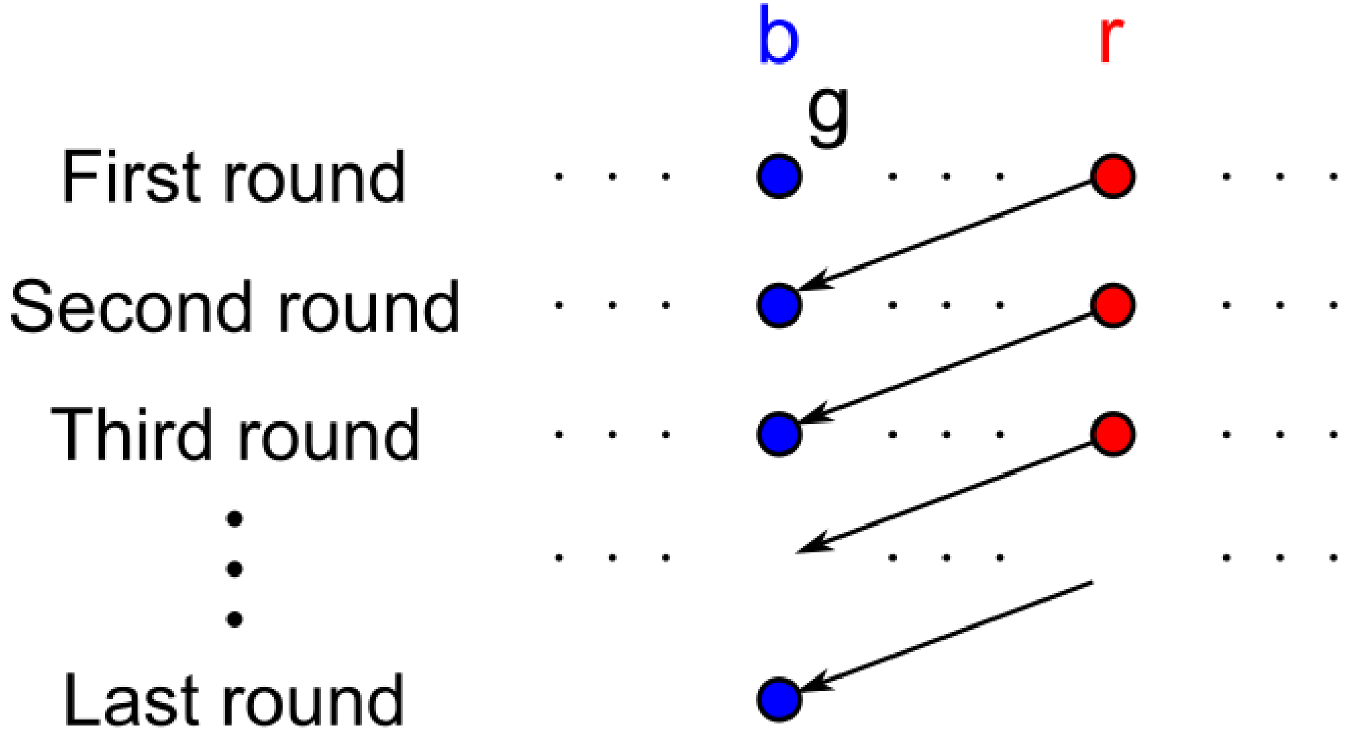

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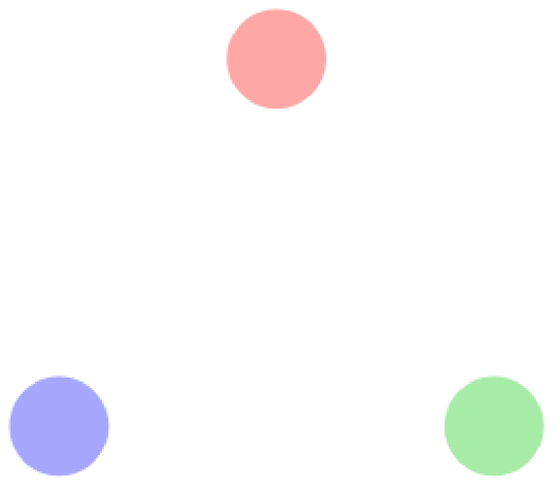

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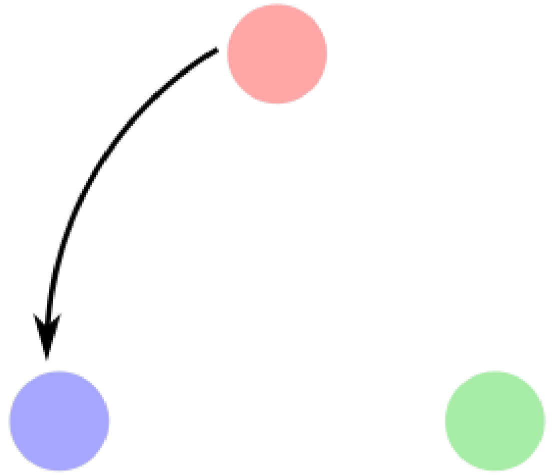

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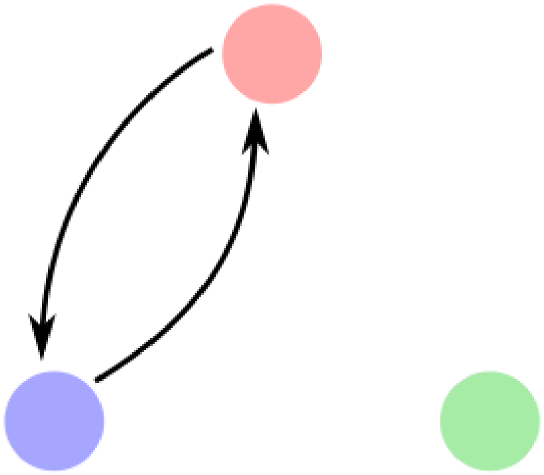

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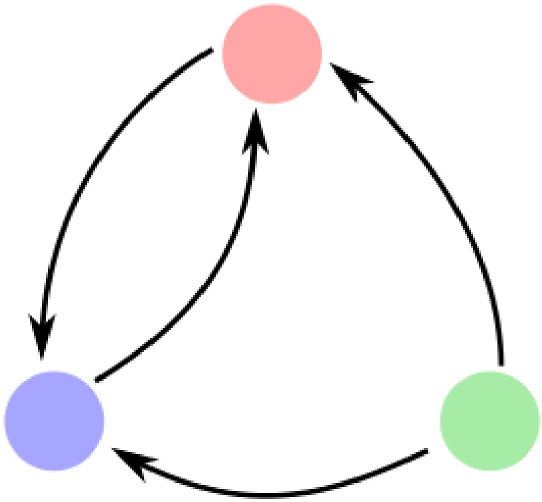

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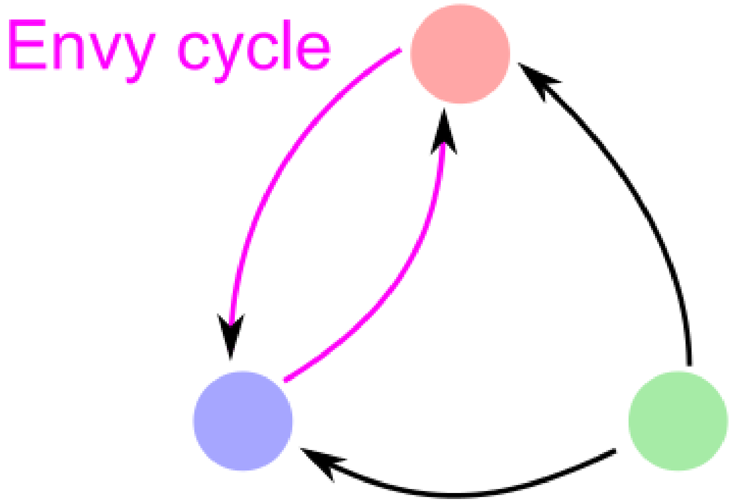

Envy-cycle elimination algorithm

[Lipton, Markakis, Mossel, and Saberi, *EC* 2004]

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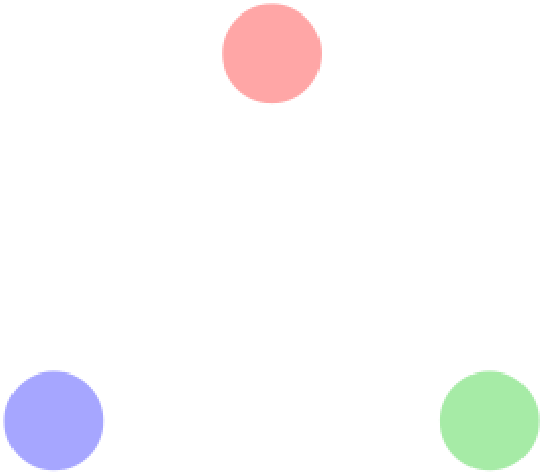

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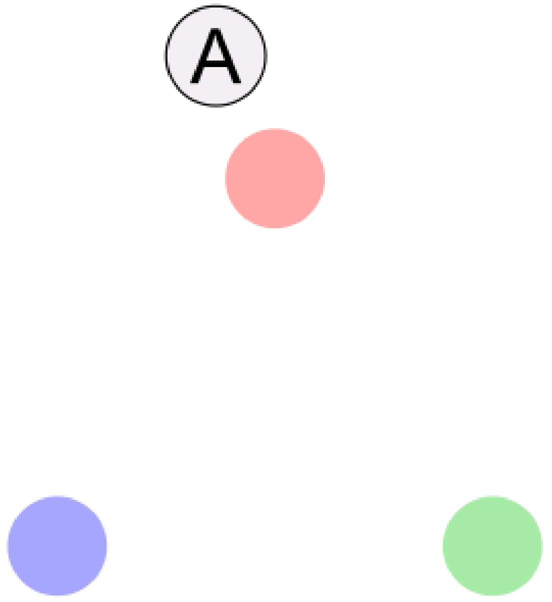

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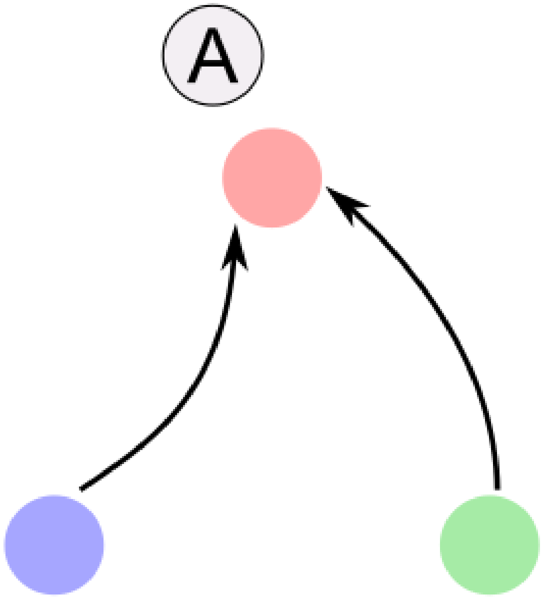

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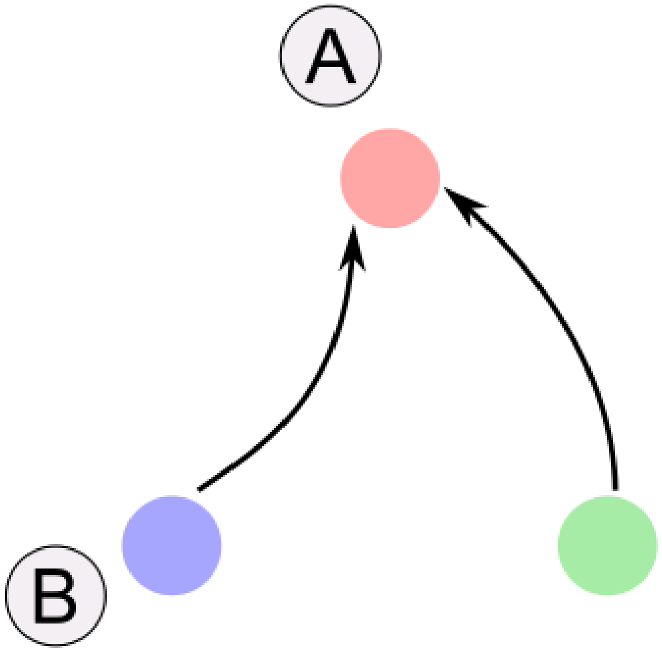

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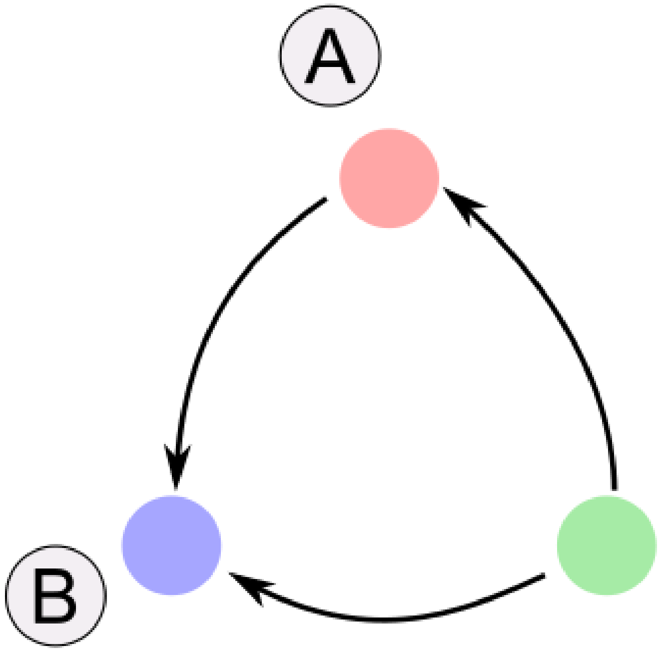

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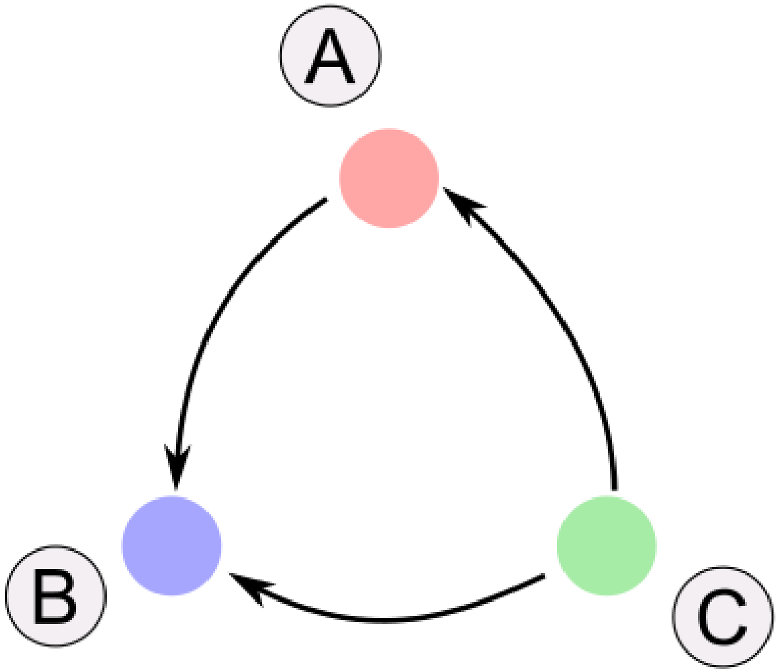

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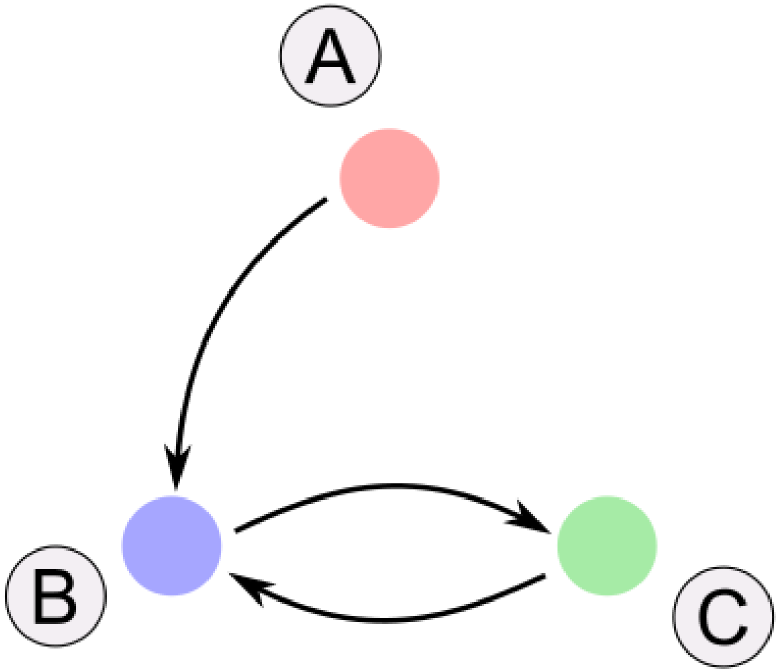

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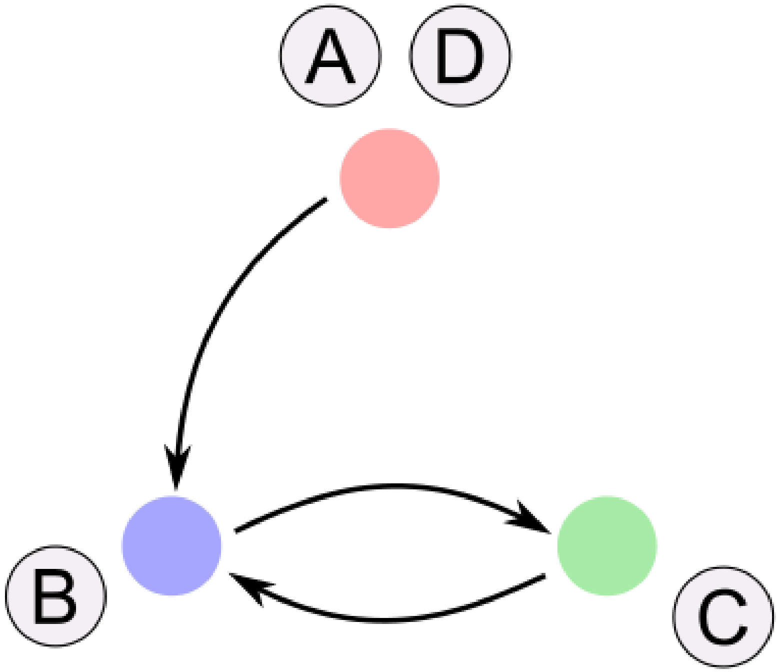

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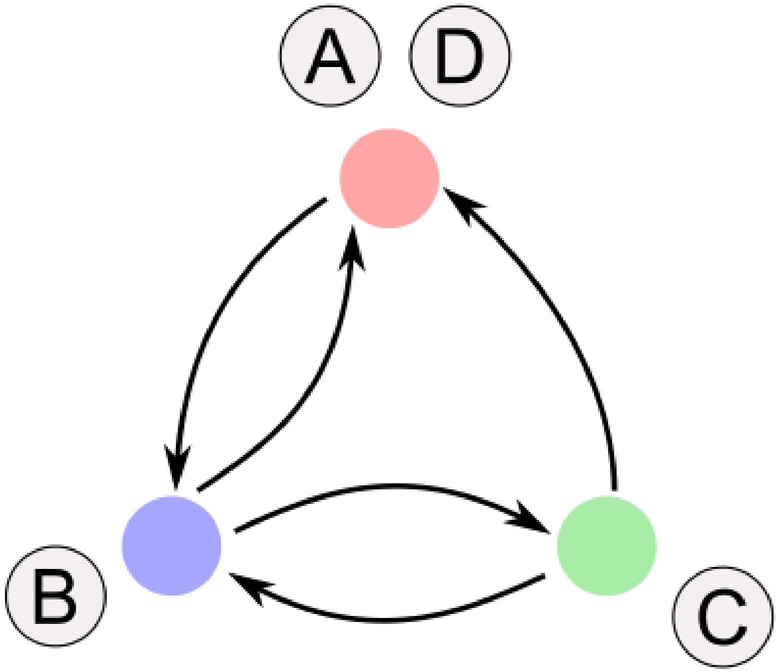

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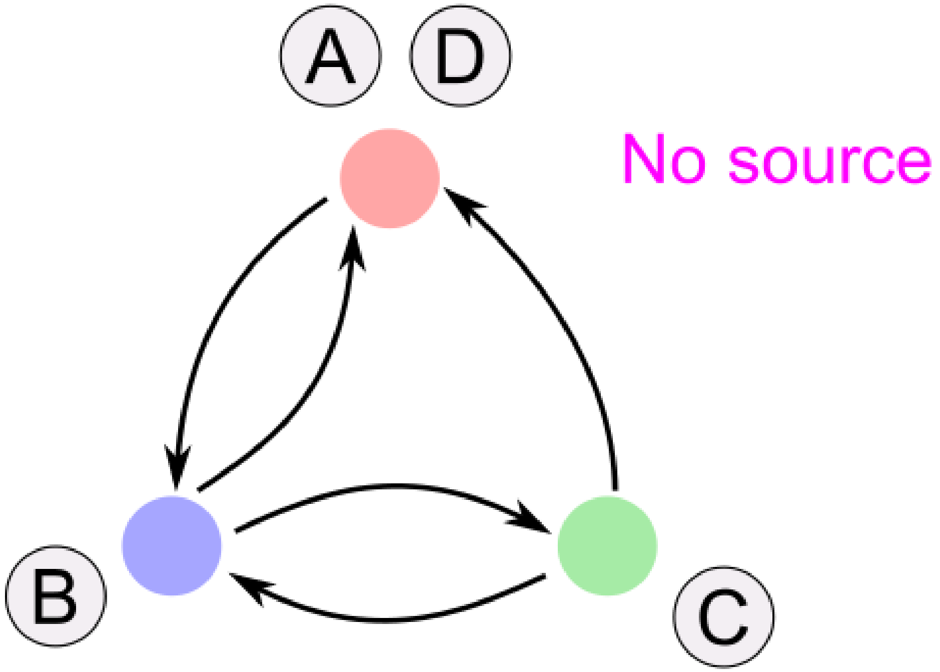

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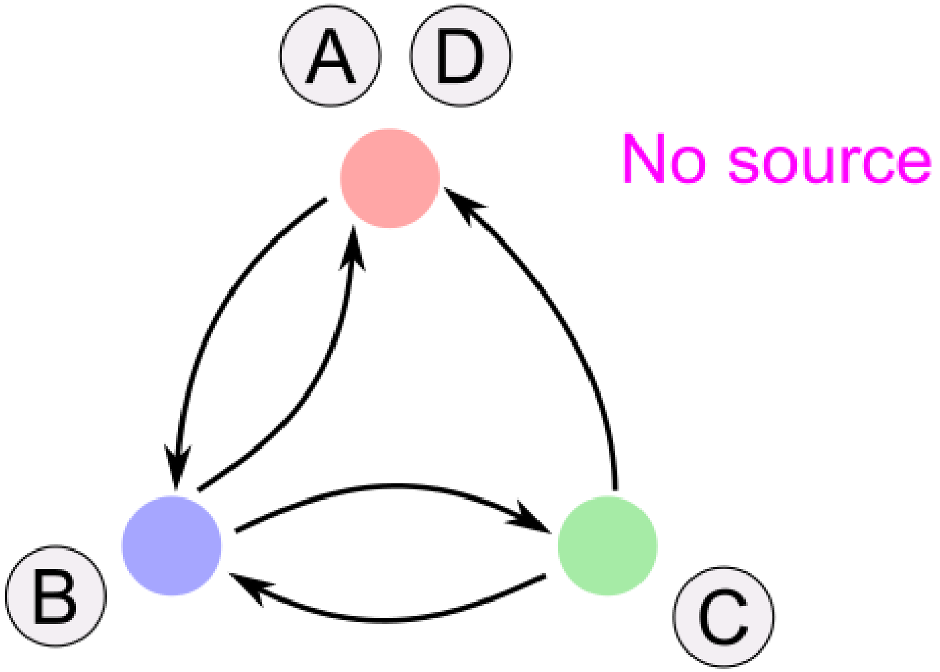

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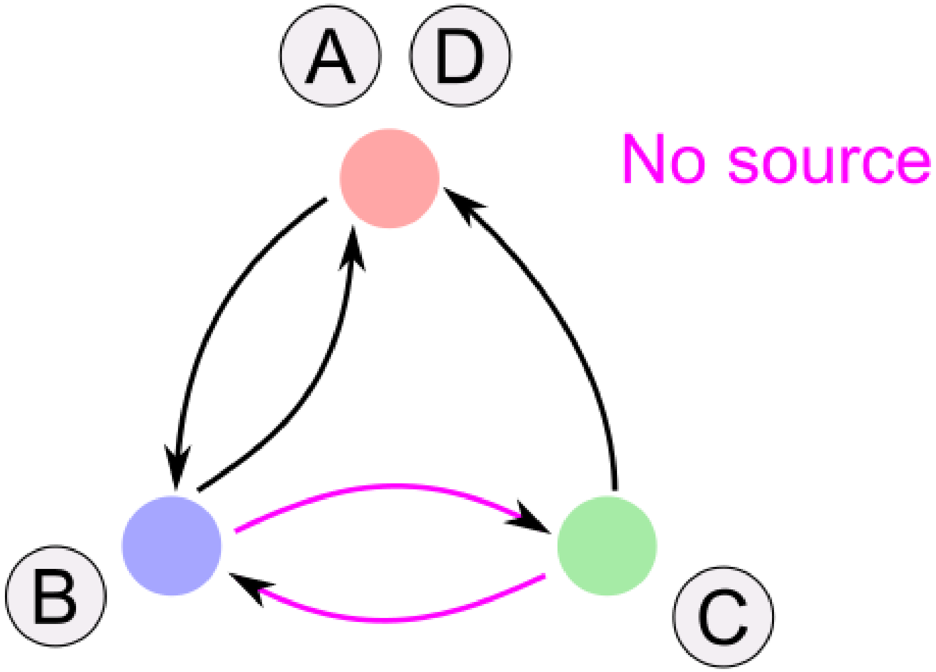

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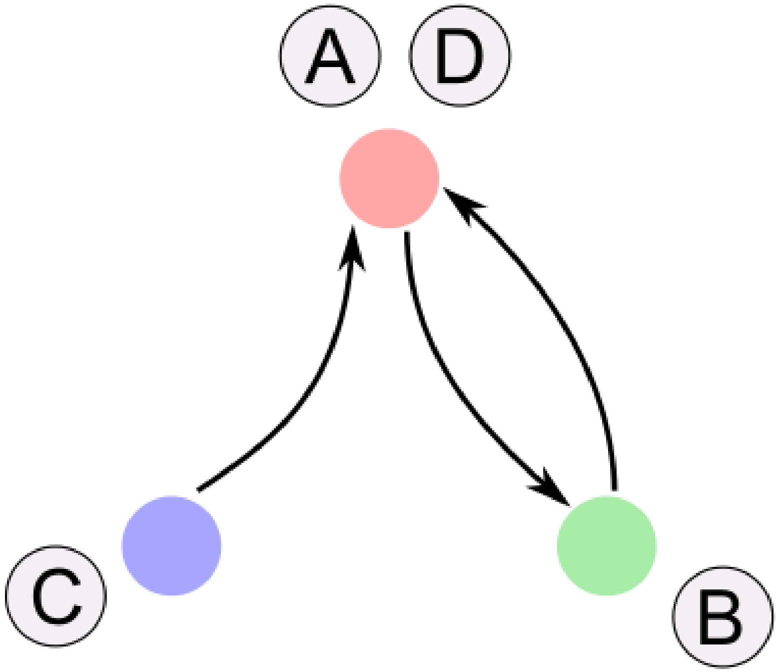

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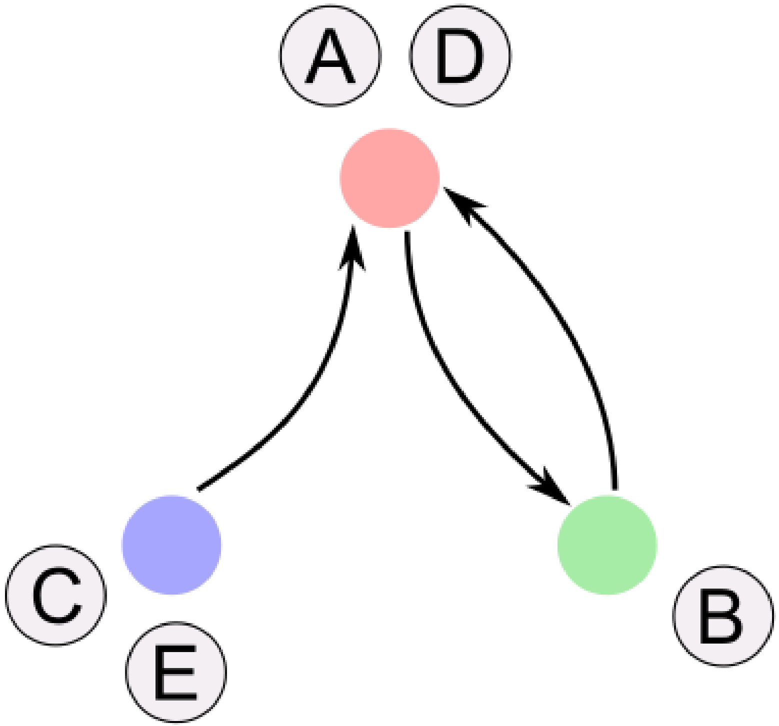

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- With n agents, at most $O(n^2)$ cycle resolutions required to create a source.
- Polynomial running time!

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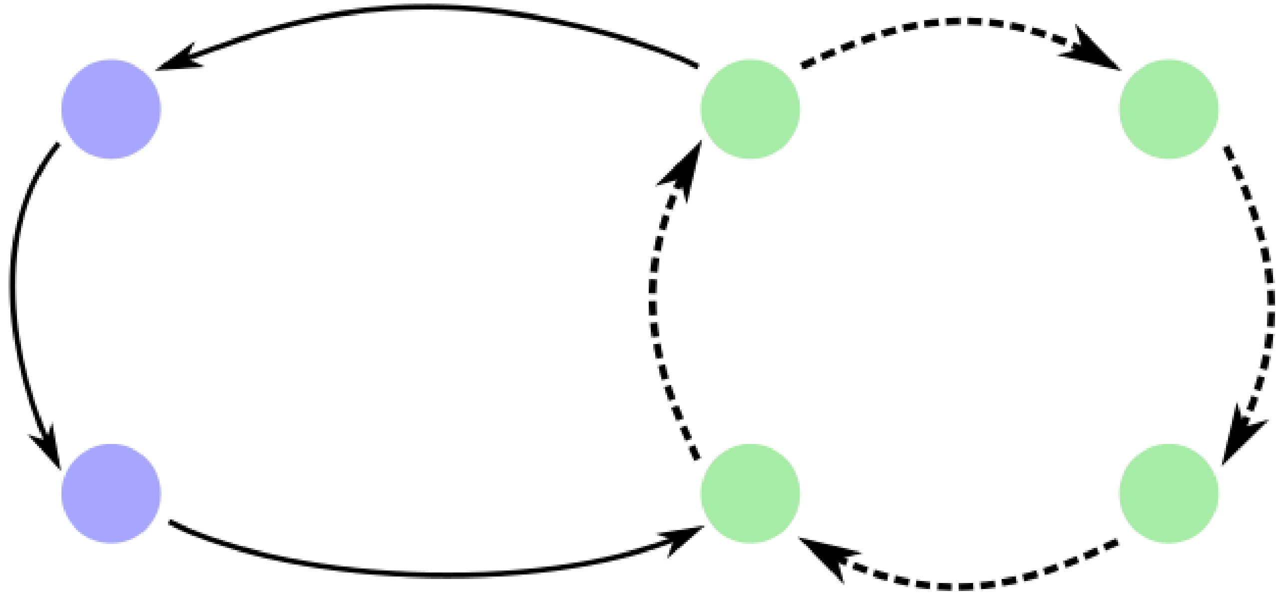

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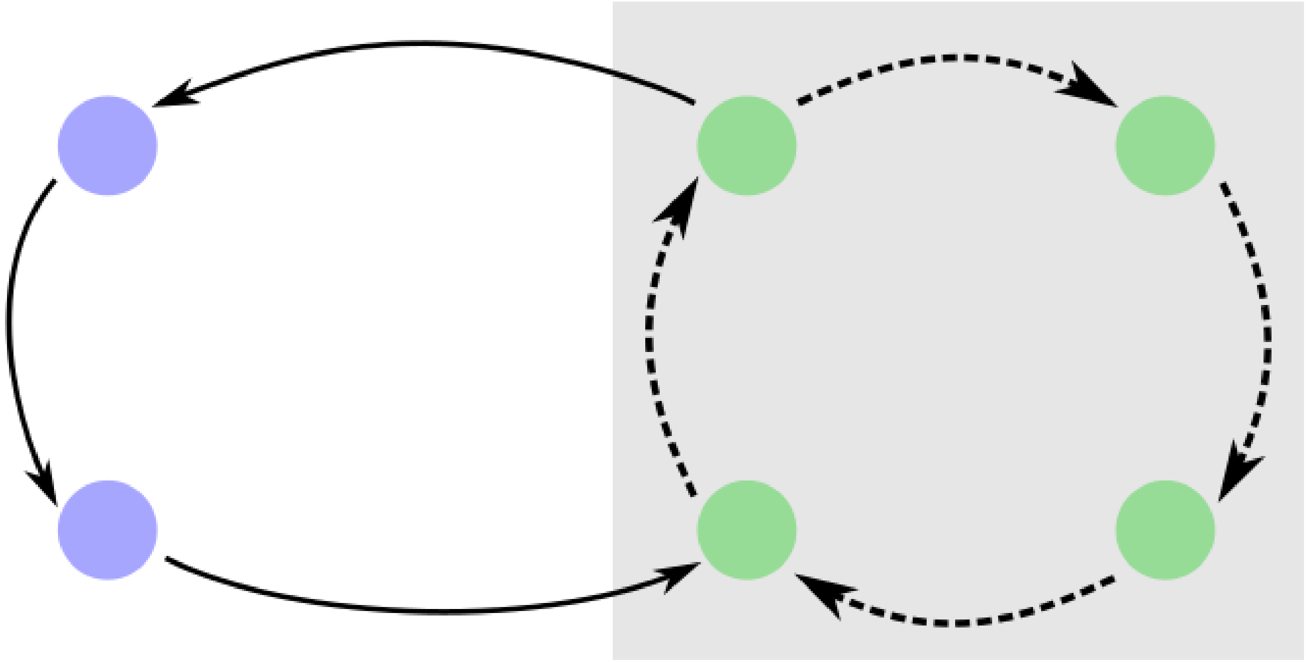

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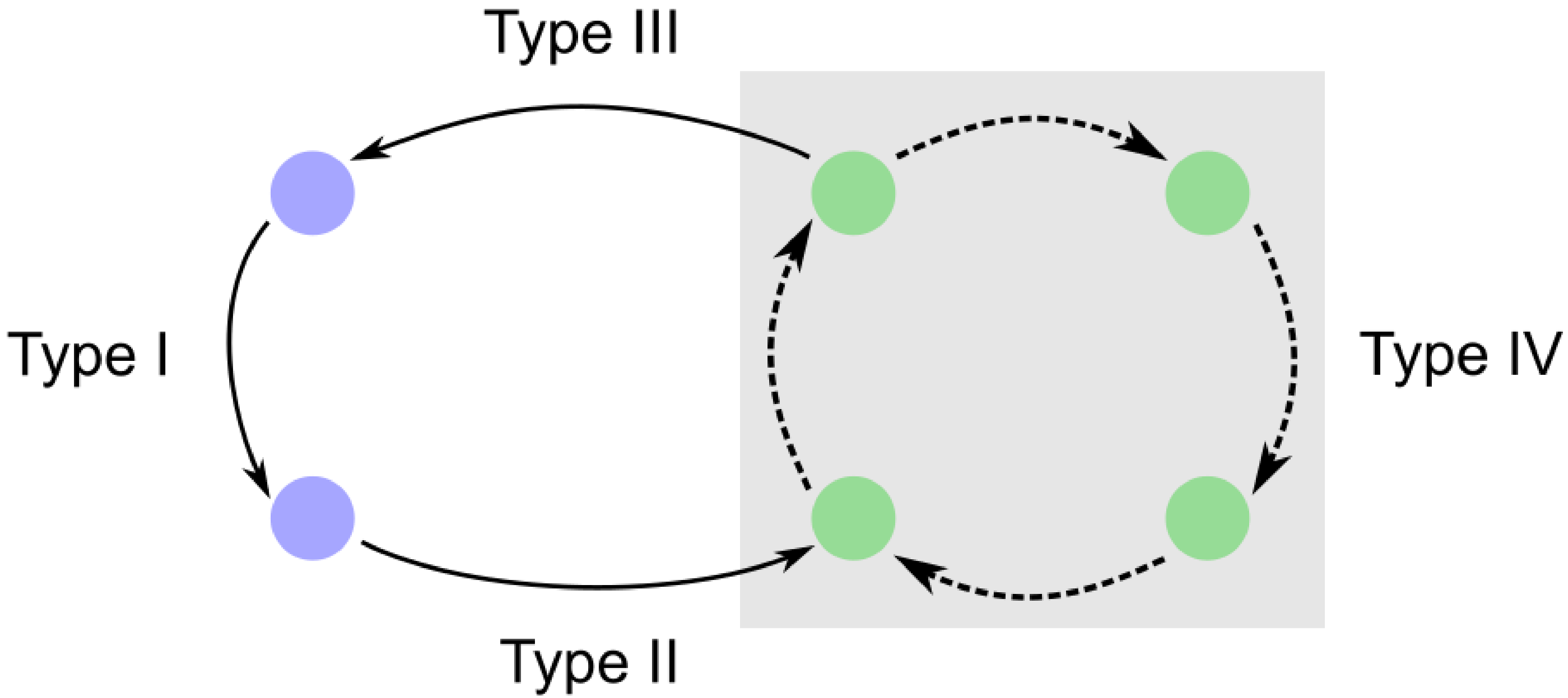

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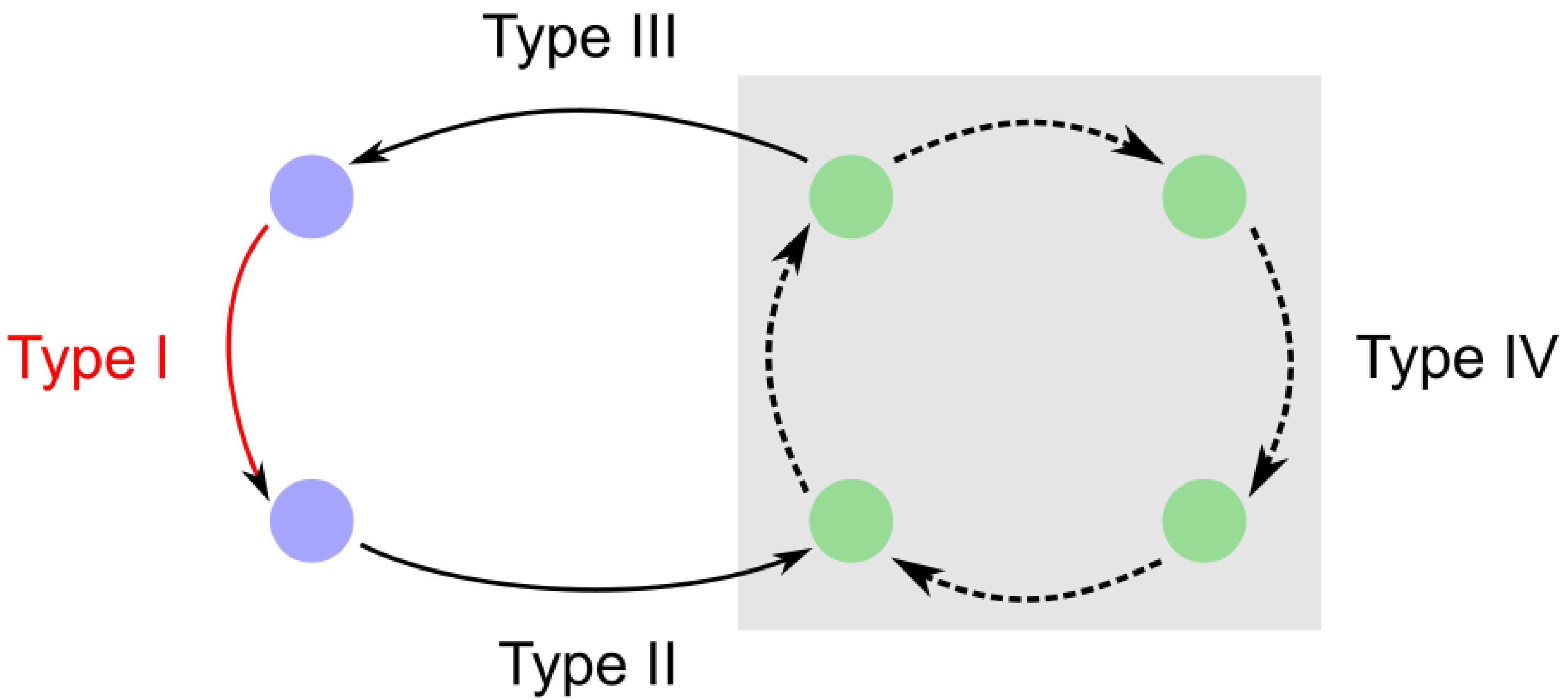

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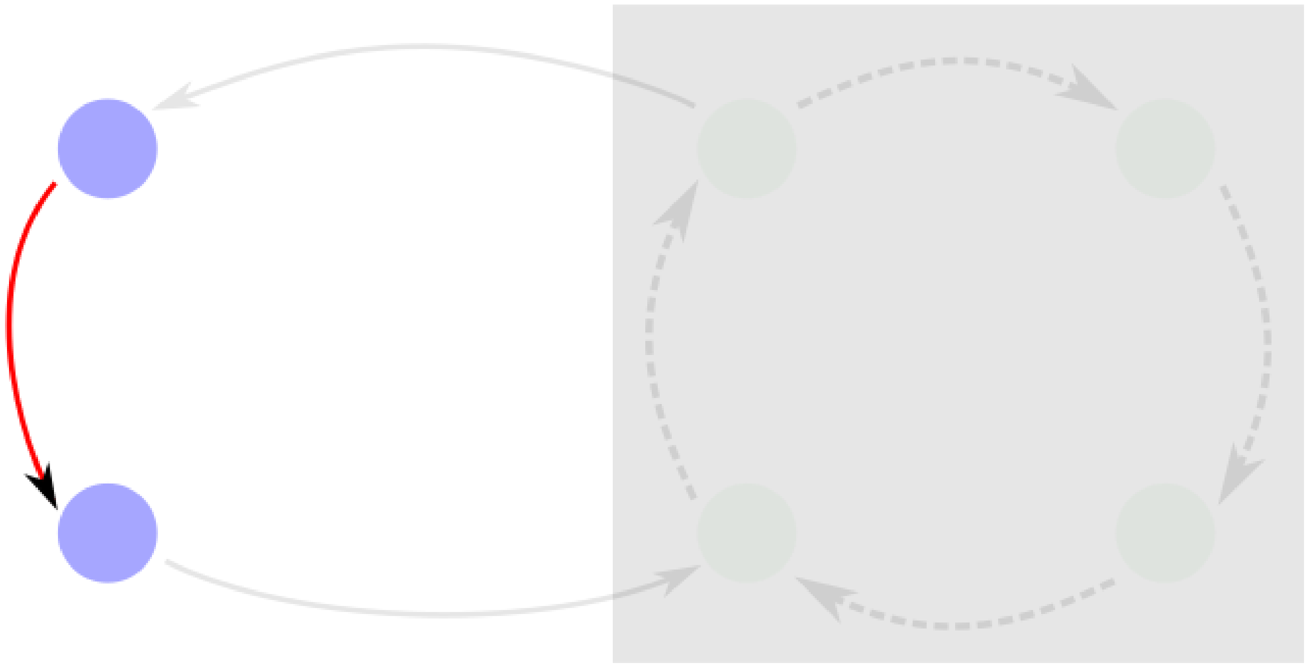

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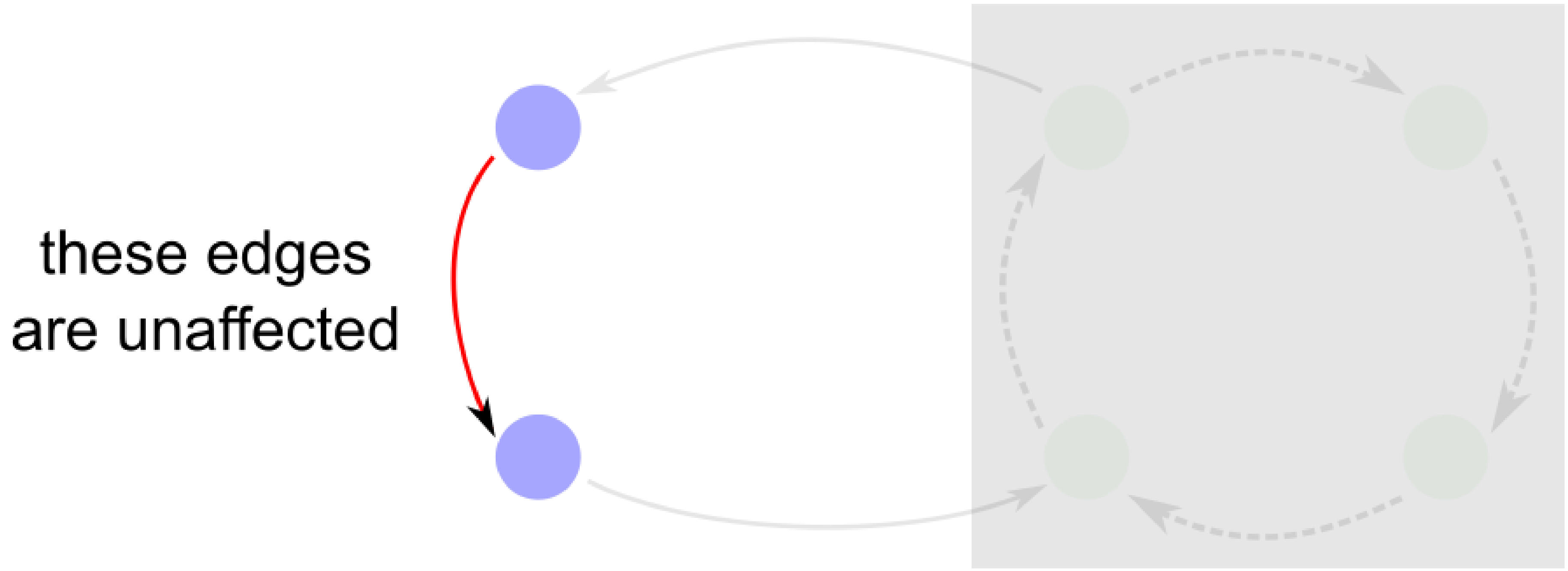

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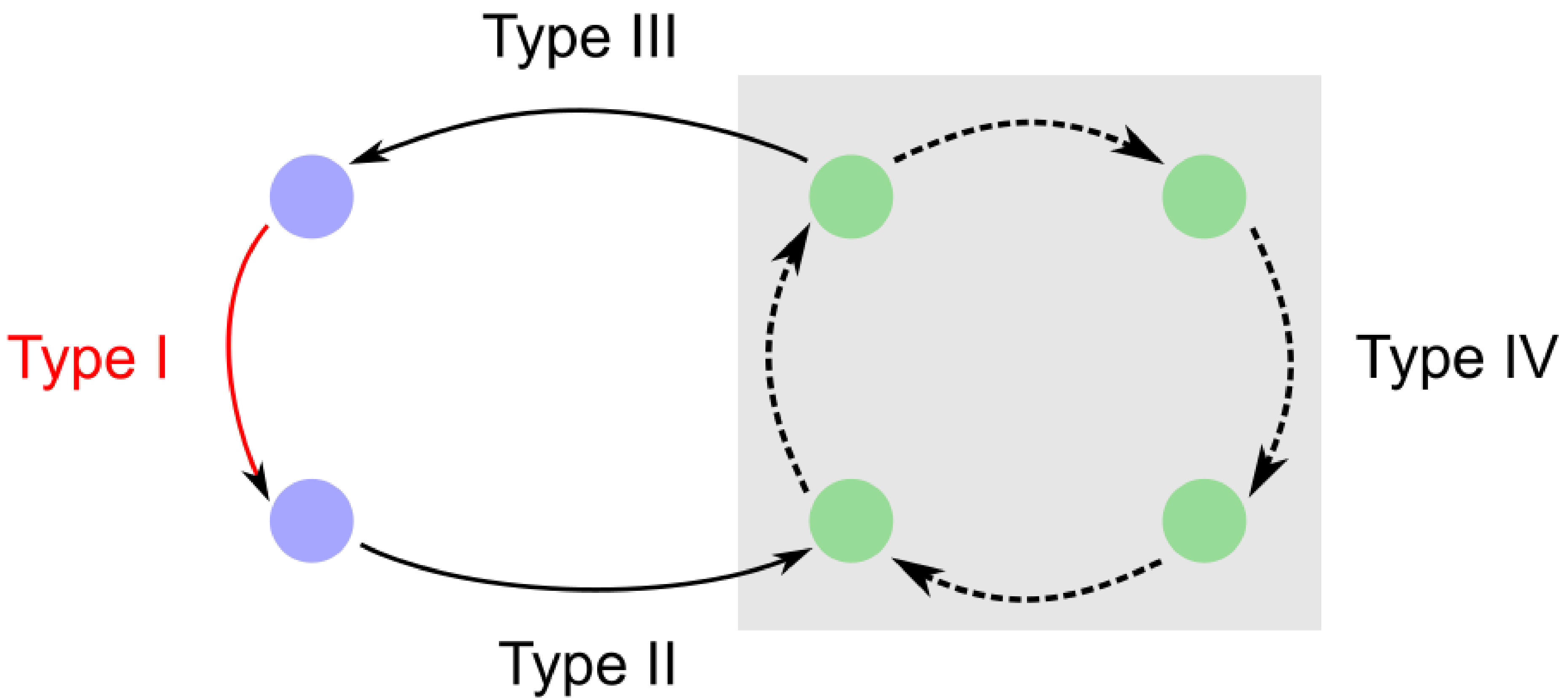

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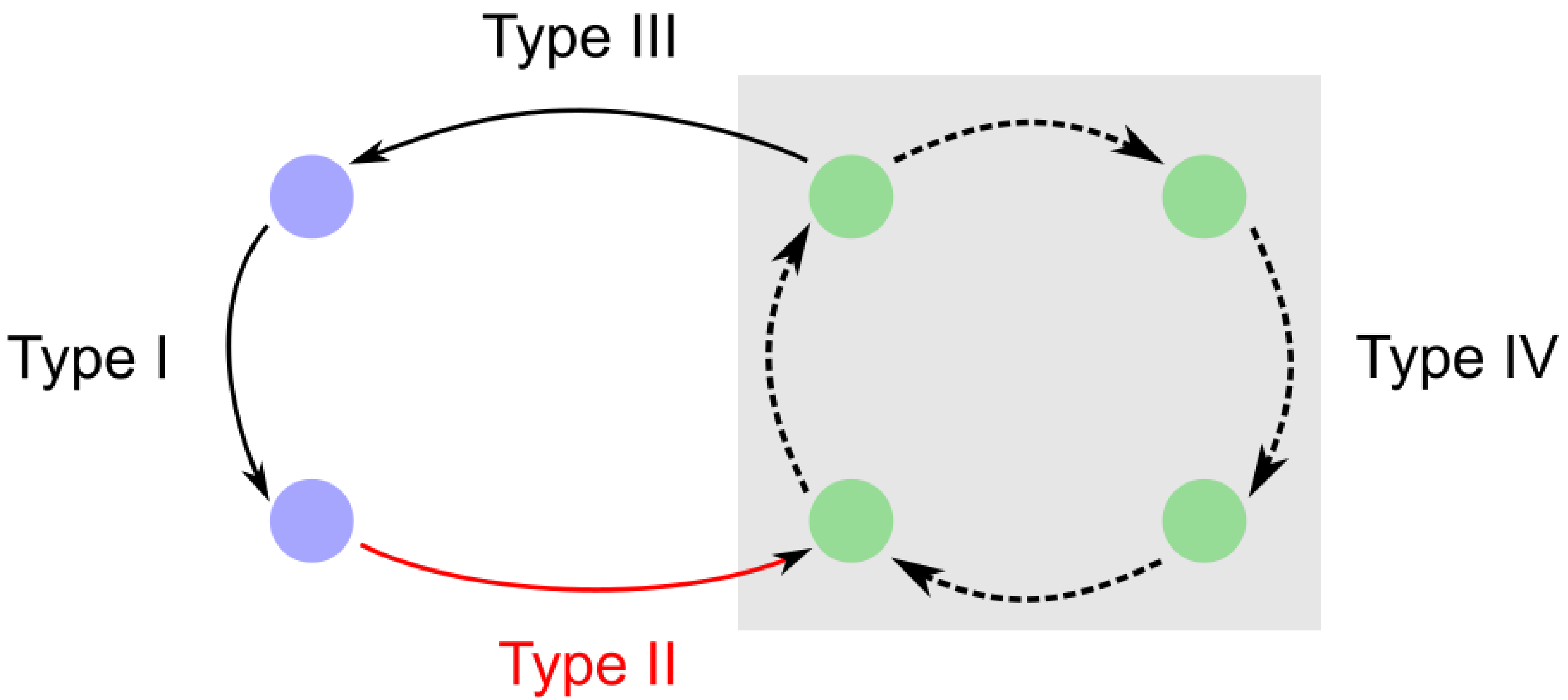

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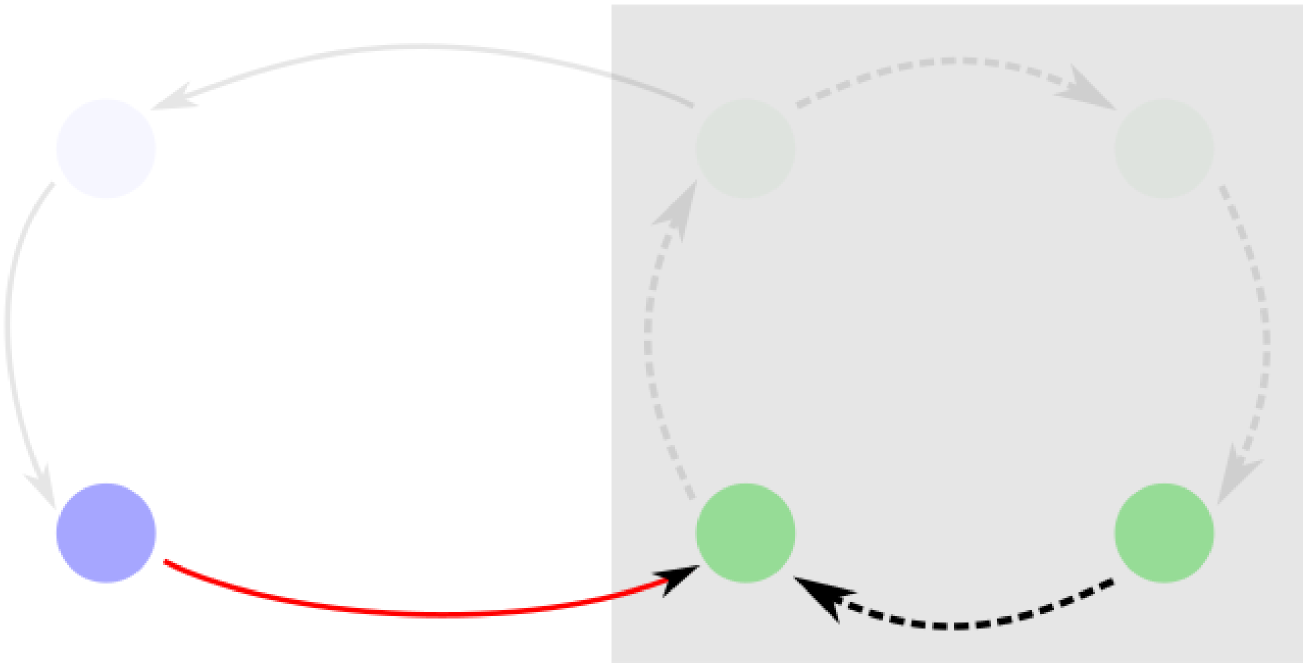

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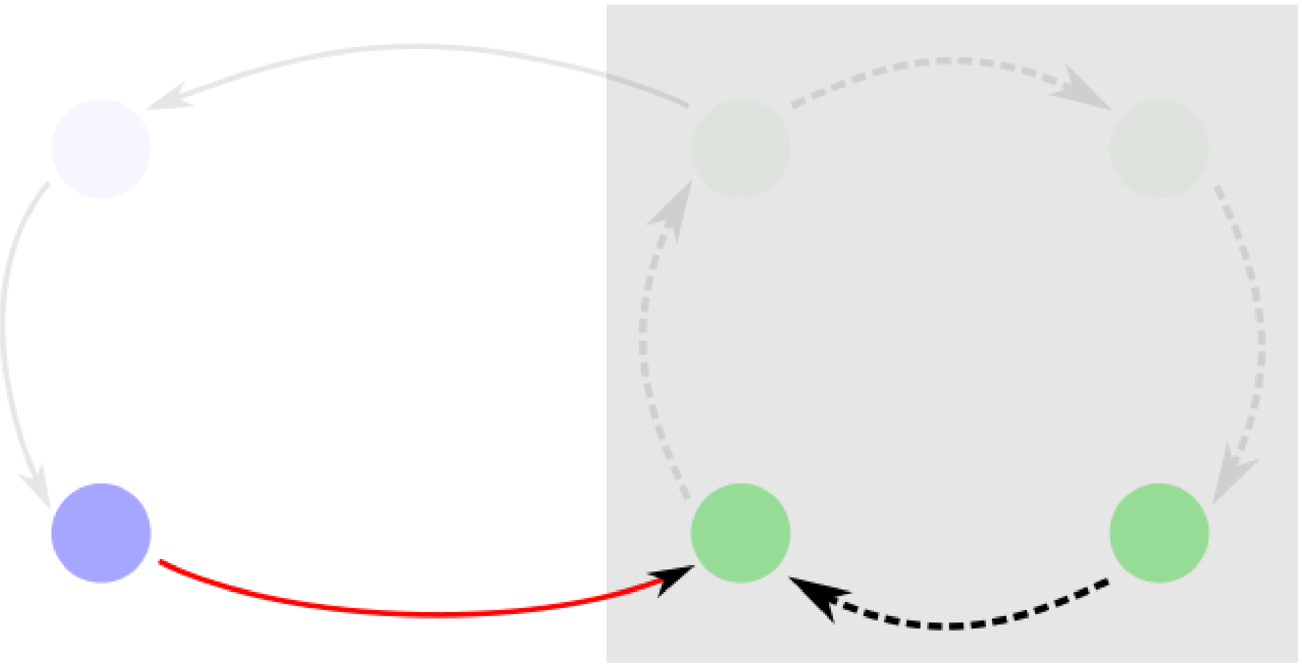

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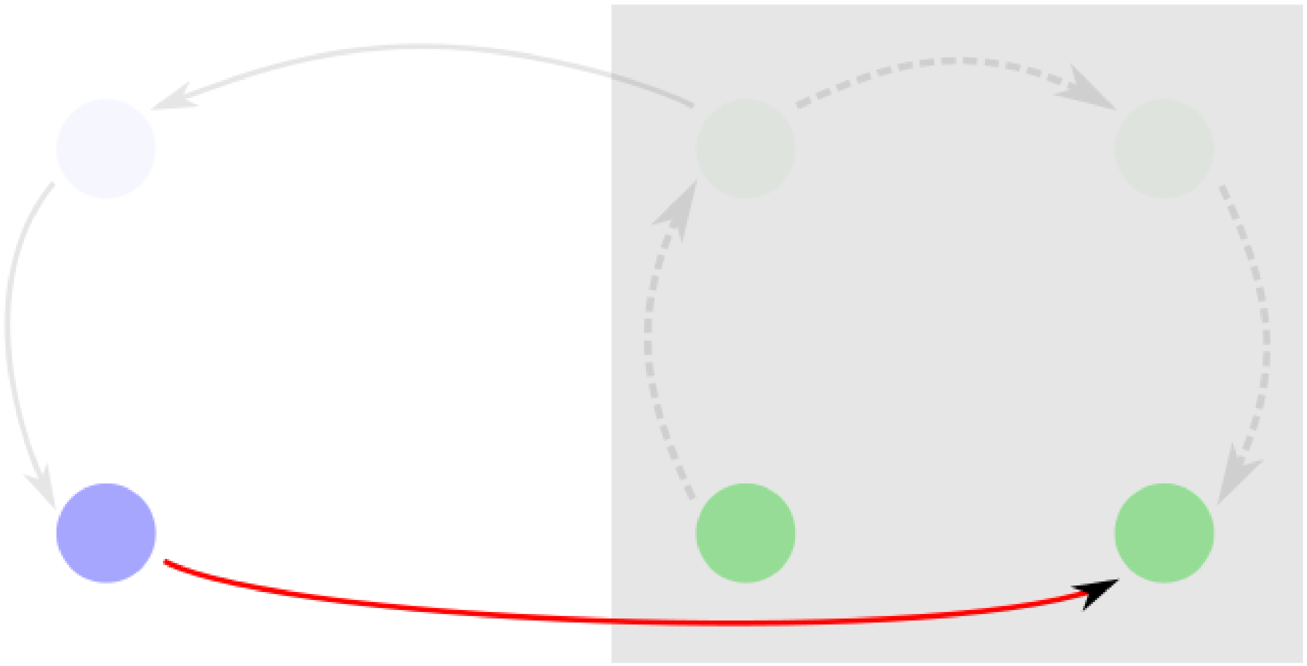

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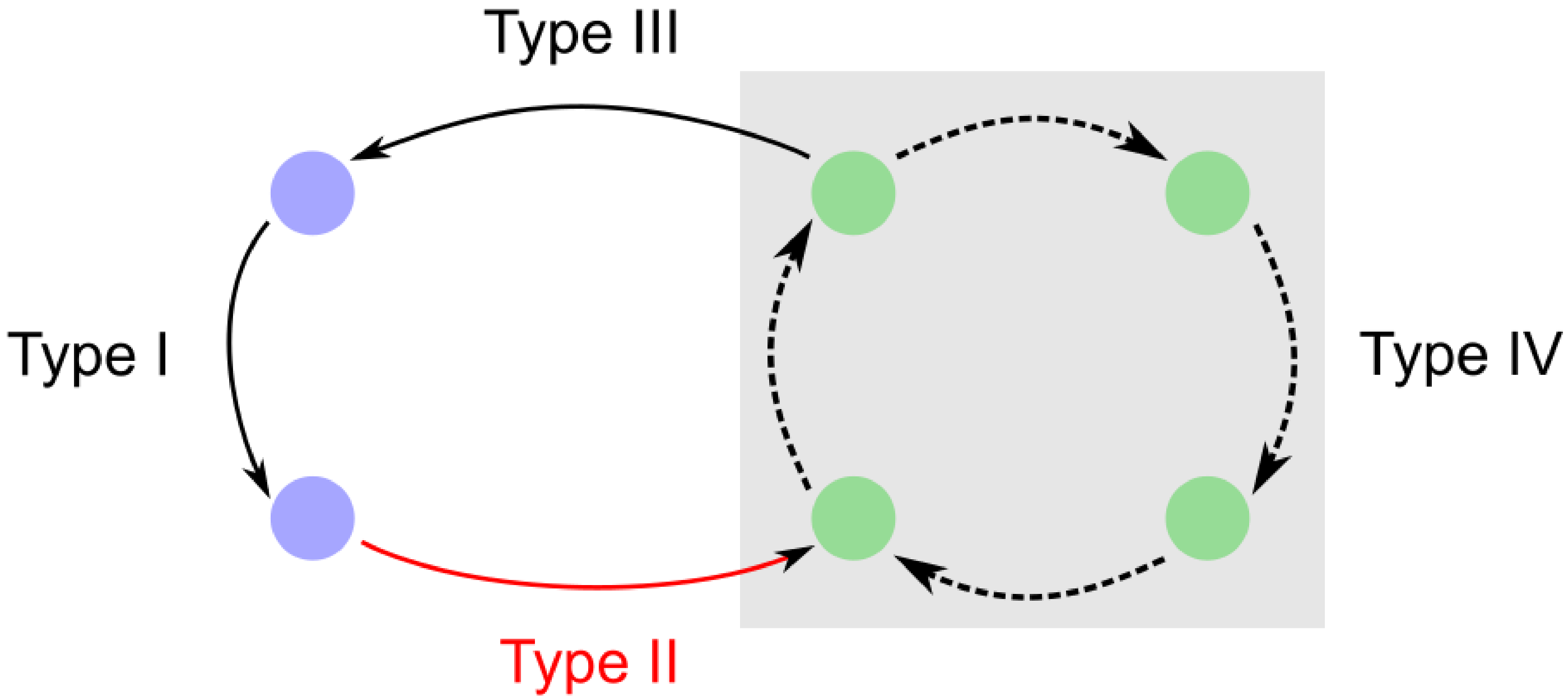

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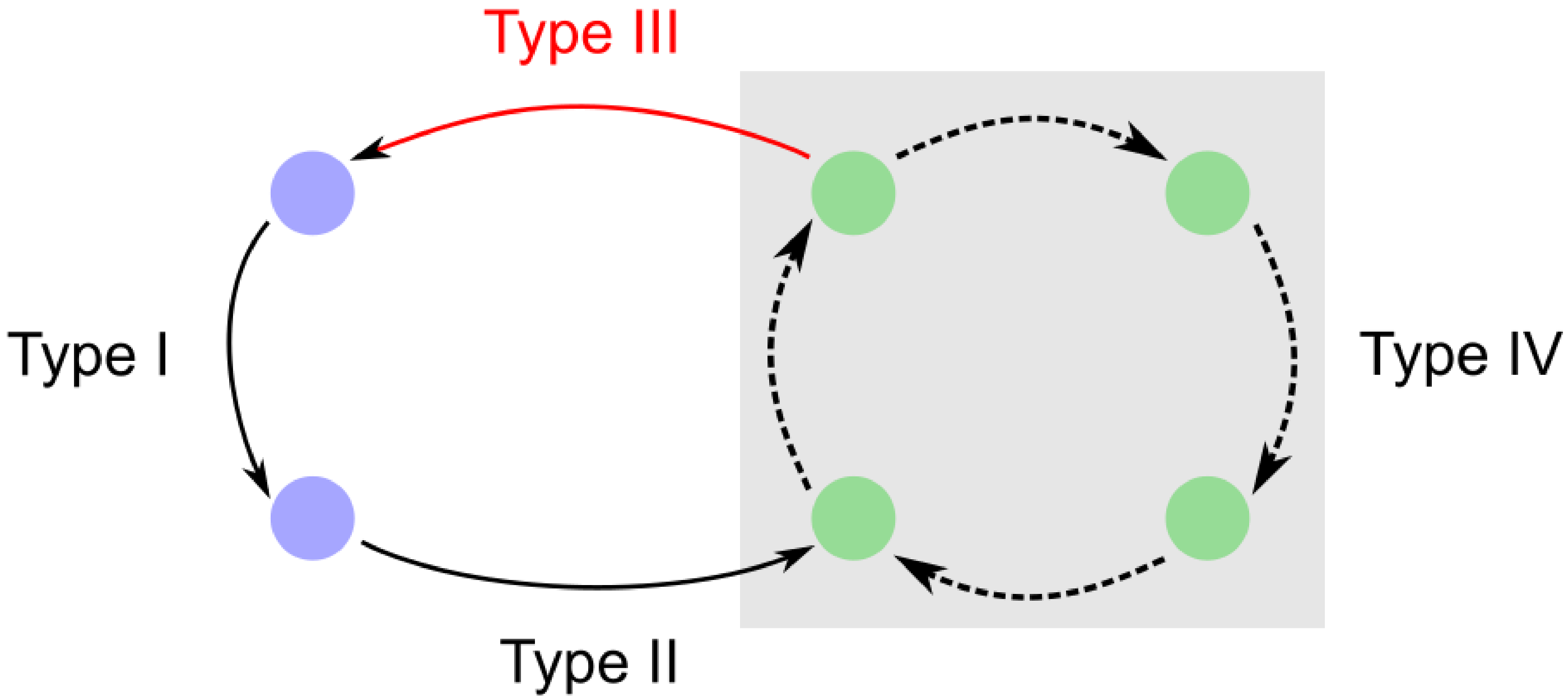

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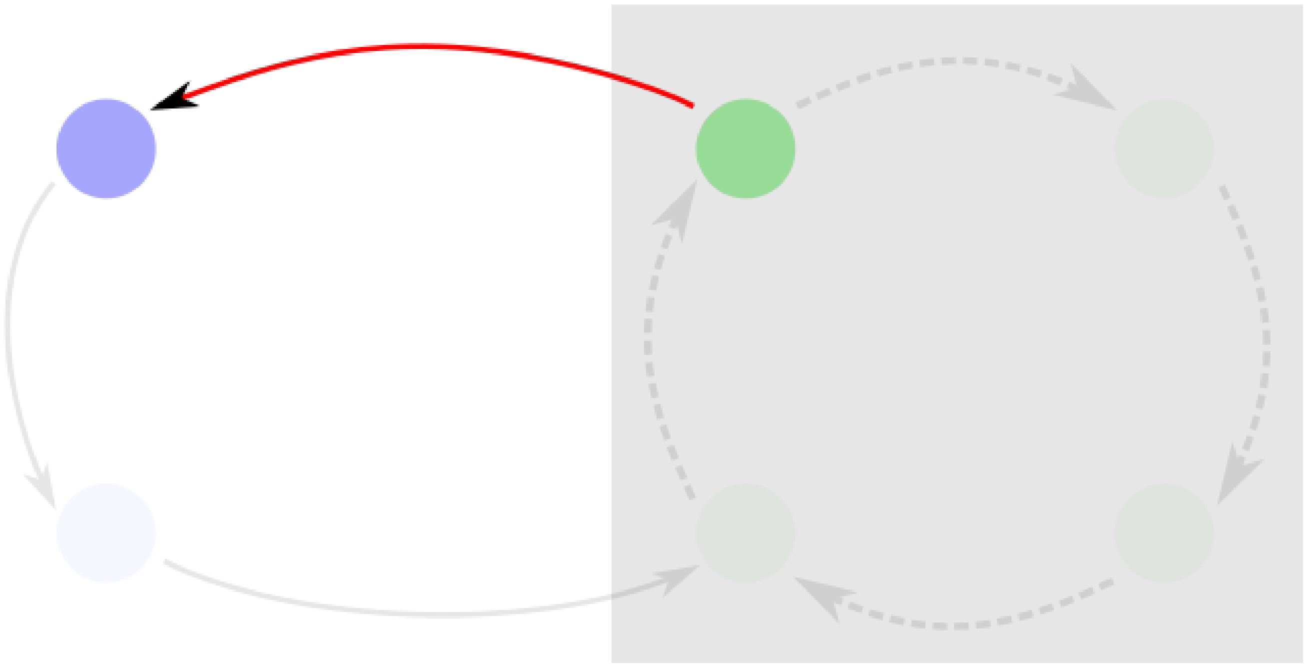

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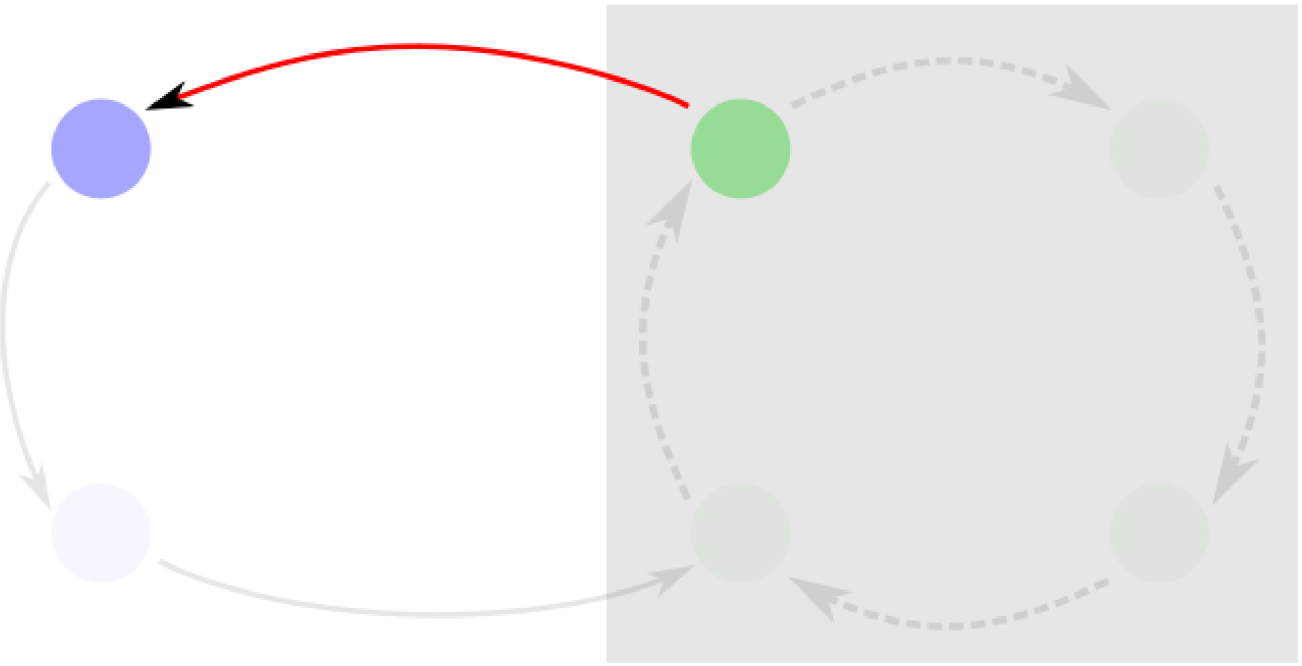

these edges can either stay or disappear (no new such edges are added)

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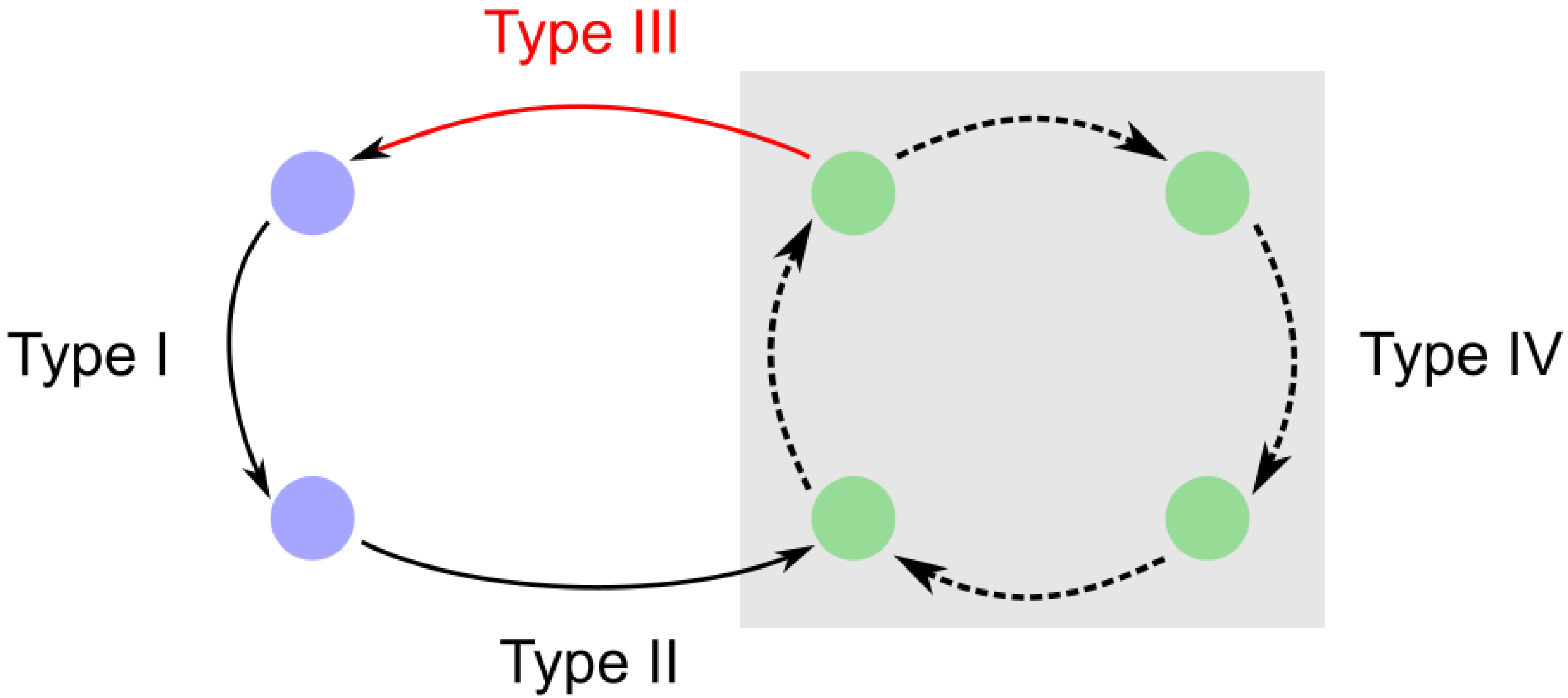

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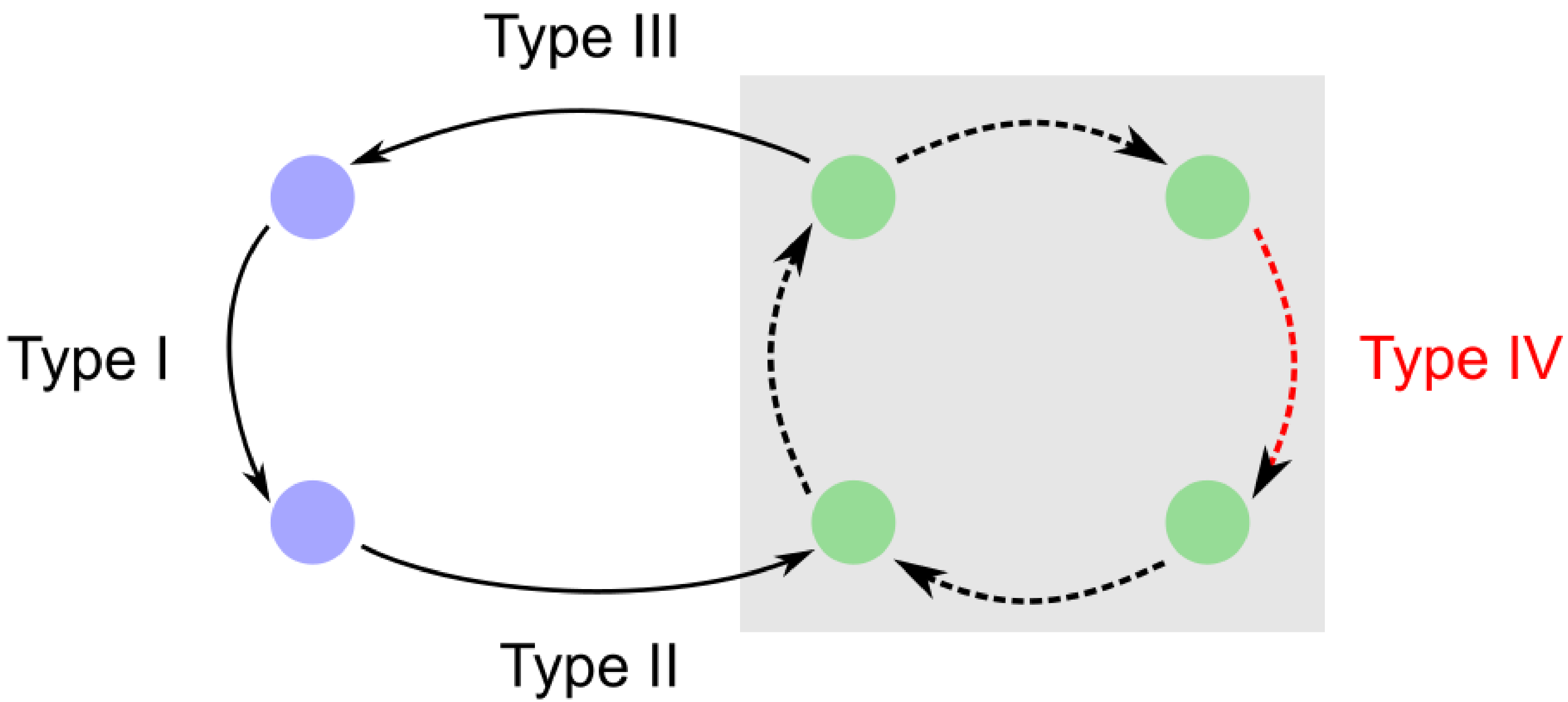

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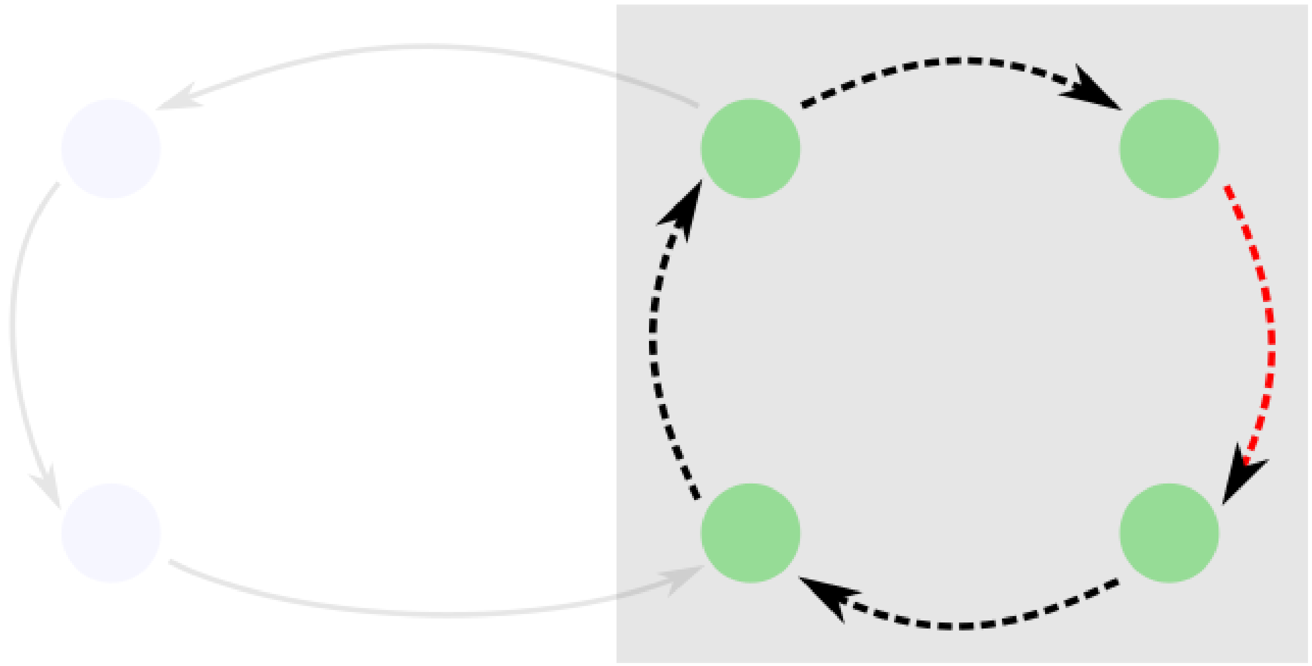

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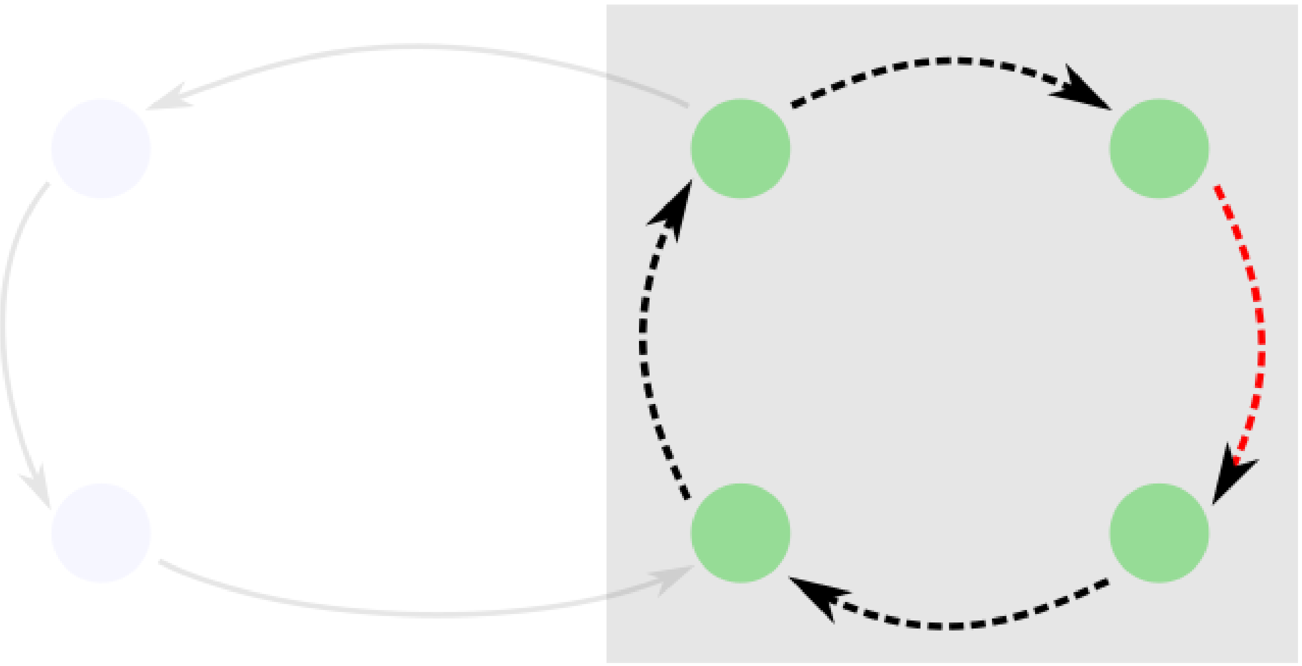

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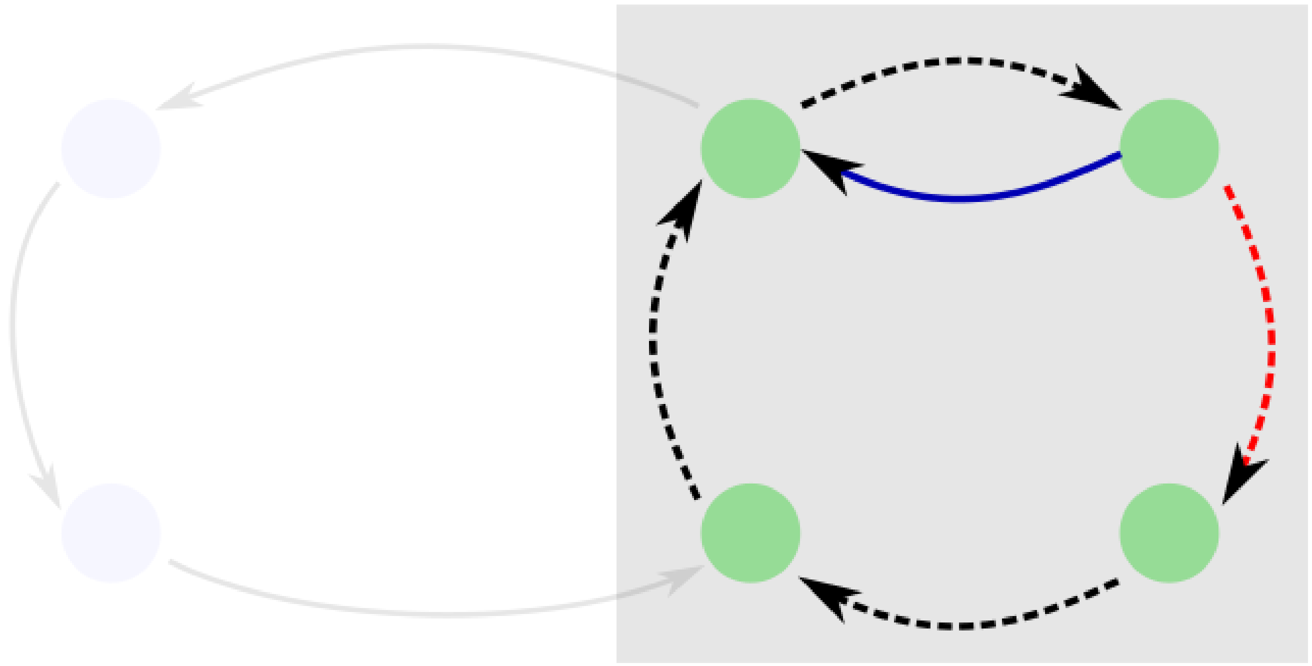

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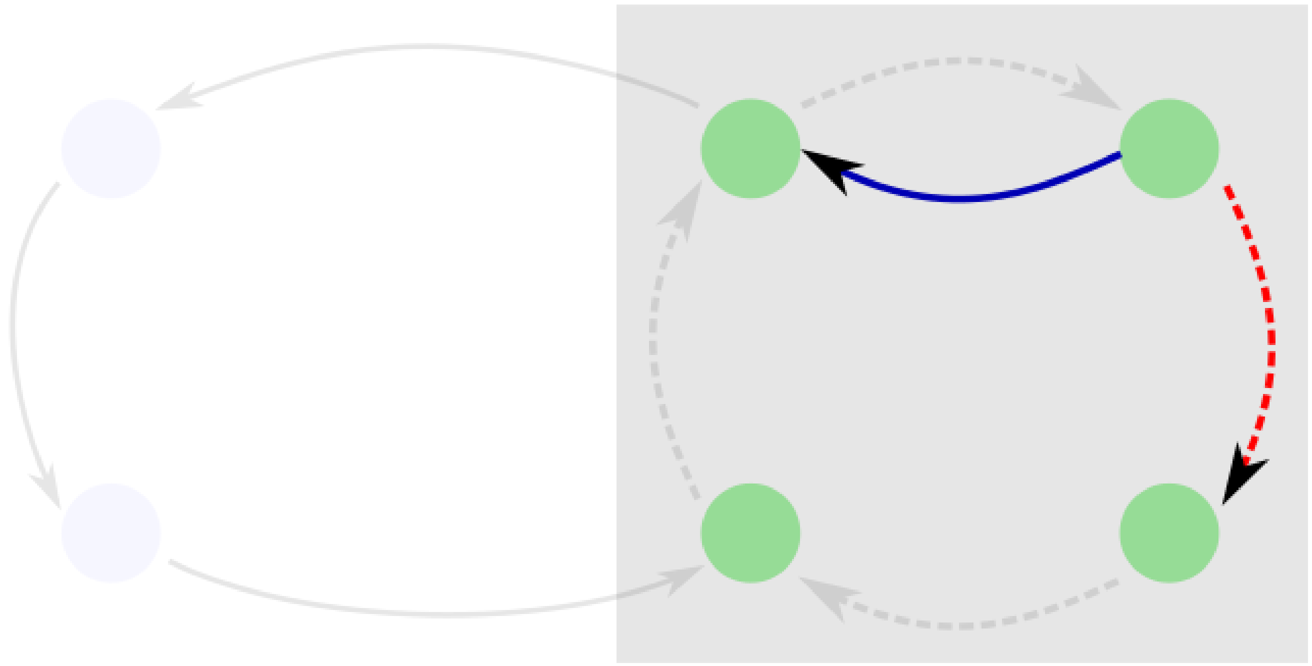

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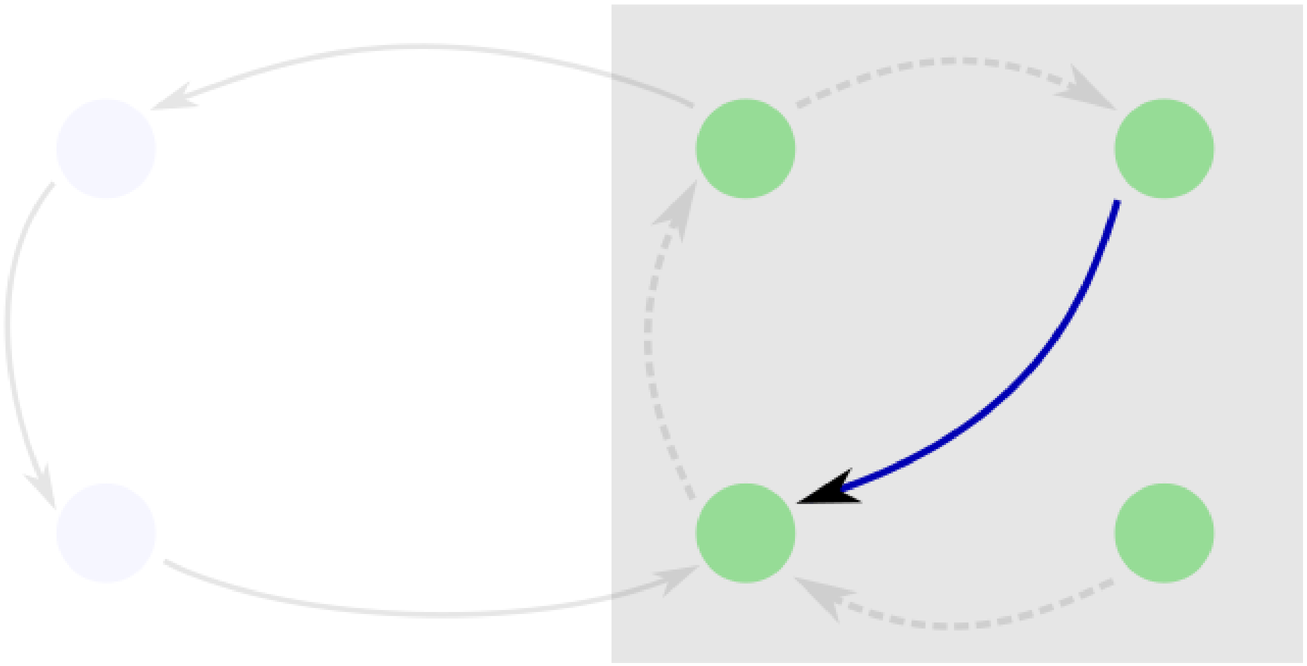

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Allocation A is EF1 if for every pair of agents i, k, there exists a good $j \in A_k$ such that $v_i(A_i) \geq v_i(A_k \setminus \{j\})$.

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> If the partial allocation at the beginning of an iteration is EF1, then the partial allocation at the end of that iteration is also EF1.

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Suppose good g is assigned to the source agent s. Then,

$$v_i(A_i) \geq v_i(A_s \cup \{g\} \setminus \{g\})$$

which means that EF1 is preserved.

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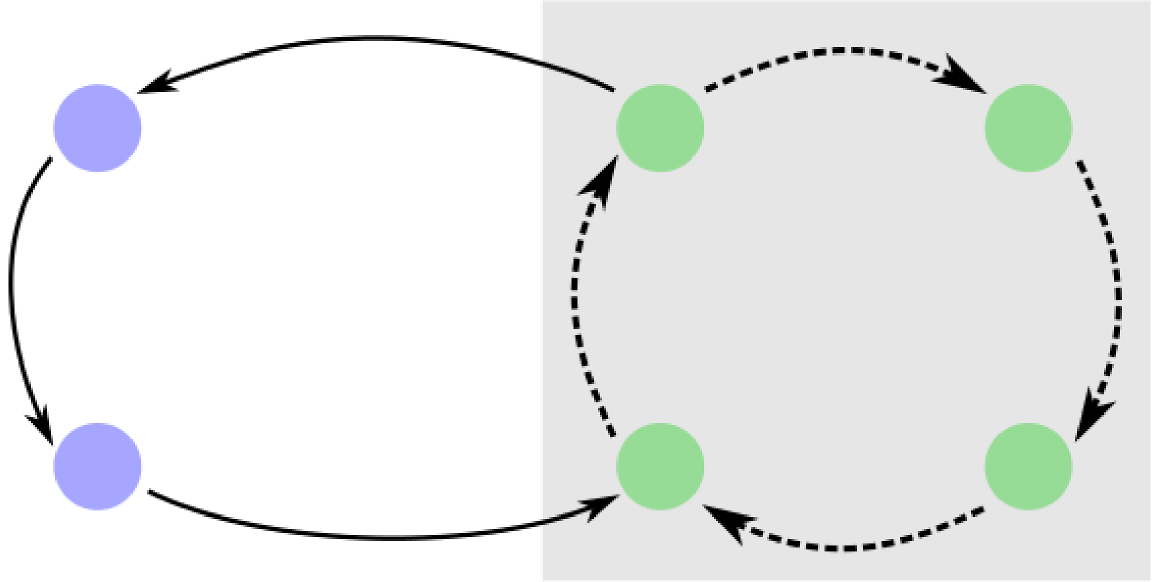

Does envy-cycle elimination algorithm return an EF1 allocation?

We will argue that each iteration of the algorithm "preserves" EF1.

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From their perspective, the bundles in the cycle are only shifted around. So, EF1 relations are the same as before.

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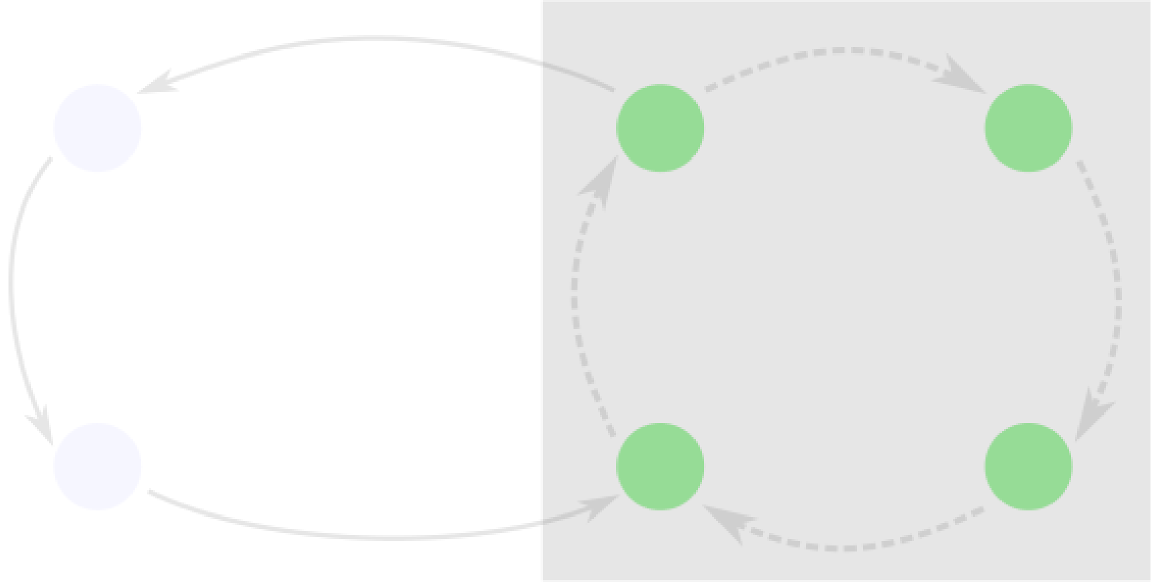

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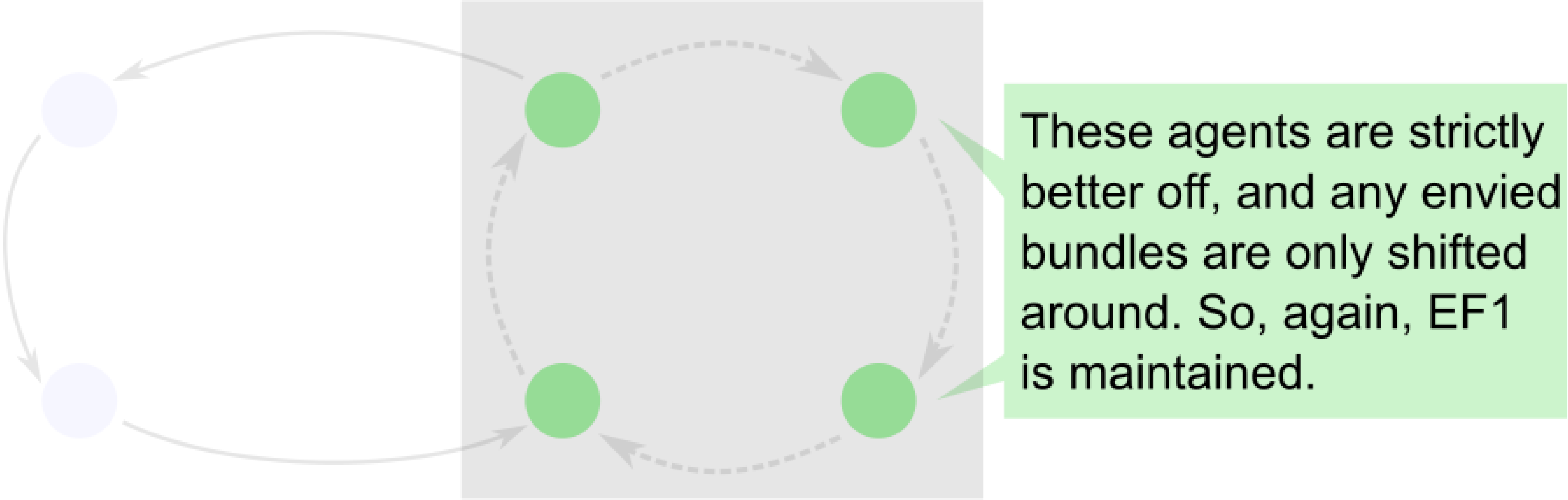

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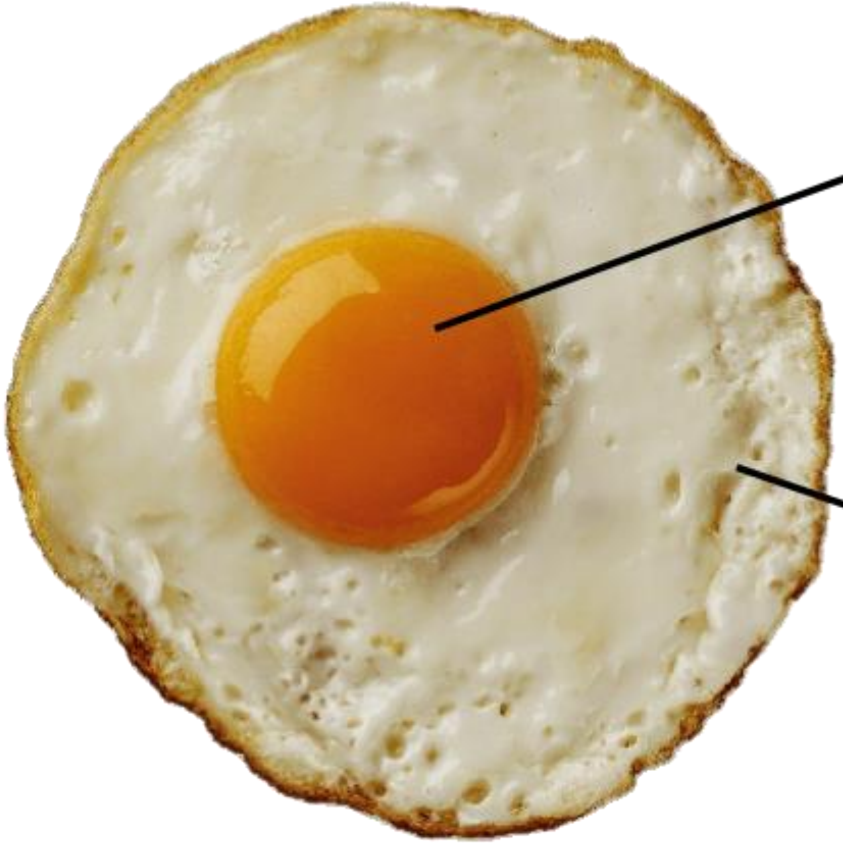

Additive: $v_i(S) = \sum_{j \in S} v_i(\{j\})$

Monotone: $S \subseteq T \Rightarrow v_i(S) \leq v_i(T)$

Reminders

Project presentations in online mode on Apr 03 (Sun).

Project reports due by Apr 10 (Sun).

Next Time

Fairness and Efficiency

References

- Envy-cycle elimination algorithm

  Richard Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi
  “On Approximately Fair Allocations of Indivisible Goods”
  EC 2004, pg 125-131
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